Supersymmetric Gödel Universes in String Theory

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Abstract

Supersymmetric backgrounds in string and M-theory of the Gödel Universe type are studied. We find several new Gödel Universes that preserve up to 20 supersymmetries. In particular, we obtain an interesting Gödel Universe in M-theory with 18 supersymmetries which does not seem to be dual to a pp-wave. We show that not only T-duality but also the type-IIA/M-theory S-duality can give supersymmetric Gödel Universes from pp-waves. We find solutions that can interpolate between Gödel Universes and pp-waves. We also compute the string spectrum on two type IIA Gödel Universes. Furthermore, we obtain the spectrum of D-branes on a Gödel Universe and find the supergravity solution for a D4-brane on a Gödel Universe.

1 Introduction

Many of the great breakthroughs in string theory in the past few years are tied to studying string theory on supersymmetric backgrounds with Ramond-Ramond fields. Some of the most notable backgrounds include D-branes [1], $\mathrm{AdS}_5 \times S^5$ [2, 3] and, more recently, pp-waves with Ramond-Ramond fluxes [4, 5, 6].

Recently, a new type of supersymmetric background with Ramond-Ramond fluxes has been found. In [7] an M-theory solution of the Gödel Universe type was found to preserve 20 supersymmetries.[1] This solution has Ramond-Ramond fluxes when compactified to type IIA string theory. The fact that it is of the Gödel Universe type means that it has closed time-like curves. In [10], it was described that the principle of Holography perhaps can remedy the problem of closed time-like curves and protect the chronology in Gödel Universe backgrounds. Furthermore, in [10], it was shown that the Gödel Universe background of [7] is T-dual to a pp-wave.[2] This discovery gives a reason for the otherwise mysterious existence of supersymmetric Gödel Universes in that we can connect them to already known solutions of string theory.

Originally the Gödel universe [12] (see [13] for an excellent review) is defined as a pressure-free perfect fluid solution in General Relativity with negative cosmological constant. The four dimensional metric in polar coordinates is given by

$$ds^2 = -dt^2 + d\rho^2 + \sinh^2\rho\,(1-\sinh^2\rho)d\phi^2 - 2\sqrt{2}\sinh^2\rho\,dtd\phi + dz^2. \tag{1.1}$$

We see that we have closed time-like curves for constant ρ with $\rho > \log(1+\sqrt{2})$. Thus, the closed time-like curves cannot be arbitrarily small. The Gödel Universe is also seen to be homogenous and to have a trivial topology. On the other hand, for supersymmetric backgrounds of the Gödel Universe type in string theory and M-theory, a typical example of a metric is

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(1-\rho^2)d\phi^2 - 2\rho^2 dtd\phi, \tag{1.2}$$

as a part of the ten or eleven-dimensional space-time. Despite the different looking metric these supersymmetric solutions still share the essential properties of the original Gödel Universe. Indeed this background is also homogenous, it has trivial topology,[3] and we have closed time-like curves when $\rho > 1$.

In this paper we lay some of the ground work for a better understanding of Gödel Universes in string and M-theory. We first consider different ways of obtaining new Gödel Universes, and provide several new supersymmetric Gödel Universe backgrounds. We find that not only T-duality, but also the type-IIA/M-theory S-duality can produce Gödel Universes from pp-waves. We find the Gödel Universes $G_{2n+1} \times \mathbb{R}^{10-2n}$ in M-theory with most

[1] Supersymmetric Gödel Universes in compactified string theory have been found in [8, 9]. These solutions are not Gödel Universes when uplifted to ten-dimensional string theory.

[2] This T-duality was also discussed in [11] but the connection to pp-waves was not considered.

[3] Here we only mean the 3 dimensional part (1.2). The remaining directions in string or M-theory could be topologically non-trivial.

supersymmetry. In particular, this includes a G_{11} solution with 18 supersymmetries, which is not related to the known background by T-duality. We also find that there exist three inequivalent Gödel Universes with 20 supersymmetries. In this consideration we note a subtlety in the T-duality of [10] on the maximally supersymmetric type IIB pp-wave [4] which means that we can get two inequivalent type IIA Gödel Universes, one with 20 supersymmetries and one without supersymmetry. Furthermore, we study two one-parameter families of backgrounds which have the interesting feature that they interpolate between a Gödel Universe and a pp-wave.

We go on to study the string spectrum of the supersymmetric type IIA Gödel Universe[10] which is dual to the type IIB pp-wave with maximal supersymmetry[4]. To do this we use the fact that this pp-wave solution can be quantized in the light-cone gauge[5]. We generalize the spectrum to include the compact direction on which we T-dualize and thereby obtain the spectrum on the Gödel Universe background. We also find the spectrum for the type IIA Gödel Universe which is S-dual to the M-theory Gödel Universe of [7].

Finally, we study D-branes on type IIA Gödel Universes. We consider the boundary conditions and find the spectrum of D-branes. We also find a new supergravity solution for a D4-brane on a Gödel Universe and uplift this to a new M5-brane solution on an M-theory pp-wave.

2 Gödel Universes in String and M-theory

In this section we consider the possible supersymmetric Gödel Universes in string and M-theory. We first consider the different ways of obtaining supersymmetric Gödel Universe solutions, either by T or S-dualizing a supersymmetric pp-wave solution, or by directly looking for solutions to the equations of motion and the Killing spinor equation. Then we go on to consider various new supersymmetric Gödel Universes and in particular we present the Gödel Universes $G_{2n+1} \times \mathbb{R}^{10-2n}$ of M-theory for $n = 1, 2, 3, 4, 5$ with most supersymmetry.

2.1 Gödel Universes from pp-waves by T-duality

As shown in [10] it is possible to find Gödel Universe backgrounds of type IIA/B string theory by T-dualizing type IIA/B pp-wave solutions. We explain here how this works in general since we use this transformation repeatedly in the following.

Consider here a pp-wave metric of the form

$$ds^2 = -2dx^+dx^- - \beta^2 \sum_{k=1}^{n} a_k^2 \left[(\tilde{x}^{2k-1})^2 + (\tilde{x}^{2k})^2 \right] (dx^+)^2 + \sum_{i=1}^{2n} (d\tilde{x}^i)^2 + \sum_{i=2n+1}^{8} (dx^i)^2, \tag{2.1}$$

with $a_k \neq 0$. Our light-cone coordinates are defined by $x^+ = t + y$ and $x^- = (t - y)/2$.

We assume here and in the following that we do not have NS-NS flux turned on in this pp-wave background. In Section 2.4 we give an example where the NS-NS flux can hinder that we get a Gödel Universe after T-duality.

We now do the coordinate transformation

$$\tilde{x}^{2k-1} + i\tilde{x}^{2k} = (x^{2k-1} + ix^{2k}) \exp\left(-ia_k \beta x^+\right) \quad , \quad k = 1, ..., n, \tag{2.2}$$

In the new coordinate system we have

$$ds^2 = -dt^2 + dy^2 + \sum_{i=1}^{8} (dx^i)^2 - 2\beta \sum_{i,j=1}^{2n} J_{ij} x^i dx^j (dt + dy), \tag{2.3}$$

where we defined

$$J_{2k-1,2k} = -J_{2k,2k-1} = a_k \quad , \quad k = 1, ..., n. \tag{2.4}$$

We see now that y is an explicit space-like isometry of the metric (2.3). We can therefore do a T-duality along y. This gives the metric

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{2n} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{8} (dx^i)^2 + dy^2, \tag{2.5}$$

and the NS-NS 2-form potential $B_{(2)}$ and 3-form field strength $H_{(3)}$

$$B_{iy} = \beta \sum_{j=1}^{2n} J_{ij} x^j \quad , \qquad H_{ijy} = -2\beta J_{ij} \quad , \quad i, j = 1, ..., 2n. \tag{2.6}$$

See Appendix A for our T-duality conventions. Clearly, the metric (2.5) is of the Gödel Universe type, i.e. it has closed time-like curves and a trivial topology, and for $n = 2$ and $n = 4$ the metric is the same as for the Gödel Universe backgrounds found in [7, 10].

For any Gödel Universe background that we consider in this paper the time-space components of the metric g_{0i}, where i runs over all spatial directions (in Cartesian coordinates), can asymptotically be written as $g_{0i} = \beta \sum_j J_{ij} x^j$. This gives the general definition of the antisymmetric matrix J_{ij}. Also, we define n as being the rank of J_{ij}, i.e. $2n$ is the minimal number of directions in which we can write the non-zero part of J_{ij}.

2.2 Gödel Universes from M-theory pp-waves by S-duality

In this section we note that we can obtain Gödel Universe backgrounds by S-dualizing M-theory pp-waves. This is thus another way than T-duality that pp-waves and Gödel Universe backgrounds are dual. In Section 2.4 we explain that the T- and S-dualities can be mapped to each other when $n \leq 3$.

Consider the M-theory pp-wave metric

$$ds^2 = -2dx^+ dx^- - \beta^2 \sum_{k=1}^{n} a_k^2 \left[(\tilde{x}^{2k-1})^2 + (\tilde{x}^{2k})^2 \right] (dx^+)^2 + \sum_{i=1}^{2n} (d\tilde{x}^i)^2 + \sum_{i=2n+1}^{9} (dx^i)^2, \tag{2.7}$$

with $a_k \neq 0$. For later convenience we define here our light-cone coordinates by $x^+ = t + u$ and $x^- = (t-u)/2$. Do now the coordinate transformation (2.2). This gives the metric

$$ds^2 = -dt^2 + du^2 + \sum_{i=1}^{9} (dx^i)^2 - 2\beta \sum_{i,j=1}^{2n} J_{ij} x^i dx^j (dt + du), \tag{2.8}$$

where J_{ij} is as in (2.4). Clearly we have an explicit space-like isometry in the u direction and can therefore get the corresponding type IIA string theory solution by compactifying this direction. We summarize the M/IIA S-duality rules in Appendix A. Using the relation

$$ds^2_{\mathrm{M}} = ds^2_{\mathrm{IIA}} + (du + A_\mu dx^\mu)^2, \tag{2.9}$$

we get the ten-dimensional metric

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{2n} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{9} (dx^i)^2, \tag{2.10}$$

along with the gauge fields $A_{(1)}$ and the two-form Ramond-Ramond (RR) field strength $F_{(2)}$

$$A_i = \beta \sum_{j=1}^{2n} J_{ij} x^j \quad , \qquad F_{ij} = -2\beta J_{ij} \quad , \quad i,j = 1, ..., 2n. \tag{2.11}$$

As we see from the metric we have thus obtained a Gödel Universe background of type IIA string theory. In other words, the strong coupling limit of the type IIA Gödel Universe is given by the pp-wave in M-theory.[4] We use this S-duality below to find new supersymmetric Gödel Universe backgrounds of string and M-theory.

2.3 Construction of M-theory Gödel Universes

An alternative approach to get supersymmetric Gödel Universes instead of using T and S-dualities is to directly find solutions of the equations of motions (EOMs) that have supersymmetry. In this section we give the necessary tools to do this for Gödel Universes in M-theory.[5] These tools make a systematic study of supersymmetric Gödel Universes possible. As we shall see in Section 2.6, this approach can provide solutions that we could not otherwise have obtained by use of dualities.

Ricci tensor for Gödel Universes

We consider a G_{2n+1} Gödel Universe with metric

$$ds^2 = -\left(dt + \sum_{i=1}^{n} c_i \rho_i^2 d\phi_i \right)^2 + \sum_{i=1}^{n} \left(d\rho_i^2 + \rho_i^2 d\phi_i^2 \right), \tag{2.12}$$

[4] Resolution of closed time-like curves by dimensional oxidation was also considered in [14, 11]. In these cases the solutions were lifted from compactified string theory to ten-dimensional string theory backgrounds.

[5] See also [15] for general conditions on supersymmetric Gödel Universe backgrounds.

where c_i are constants that characterize the metric.

First note that this metric is in polar coordinates contrary to the metric of sections 2.1 and 2.2. To put the metrics of these sections in this form we should perform the coordinate transformation

$$x^{2k-1} + ix^{2k} = \rho_k e^{i\phi_k} \quad , \quad k = 1, ..., n. \tag{2.13}$$

In the rest of this paper we shall always use this coordinate transformation when going between the Cartesian and polar coordinates.

Moreover, note that we are restricting ourselves to a special type of Gödel Universe metrics which is characterized by having $g_{t\phi_i}$ proportional to ρ_i^2 since this seems to be the right type of metrics to get supersymmetric Gödel Universes.

The non-zero components of the Ricci tensor for the metric (2.12) are

$$R^t{}_t = -2\sum_{i=1}^n c_i^2, \quad , \quad R^{\rho_i}{}_{\rho_i} = R^{\phi_i}{}_{\phi_i} = 2c_i^2, \quad , \quad R^t{}_{\phi_i} = -2c_i\rho_i^2\left(c_i^2 + \sum_{j=1}^n c_j^2\right). \tag{2.14}$$

We note that curvature scalar is

$$R = 2\sum_{i=1}^n c_i^2. \tag{2.15}$$

Using (2.14) one can write down the Einstein equations relating the background fluxes to the geometry. We conduct this for M-theory Gödel Universes in the following.

Equations of motion for M-theory Gödel Universes

The bosonic part of 11 dimensional supergravity is

$$I = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2\cdot 4!}(F_{(4)})^2\right) - \frac{1}{6}\frac{1}{2\kappa^2}\int C_{(3)}\wedge F_{(4)}\wedge F_{(4)}, \tag{2.16}$$

with $F_{(4)} = dC_{(3)}$. This gives the Einstein equations

$$R^\mu{}_\nu = K^\mu{}_\nu \quad , \quad K^\mu{}_\nu \equiv \frac{1}{2}\left(\frac{1}{3!}F^{\mu\sigma\xi\kappa}F_{\nu\sigma\xi\kappa} - \frac{1}{3}\delta^\mu_\nu\frac{1}{4!}(F_{(4)})^2\right). \tag{2.17}$$

The equations of motion and Bianchi identity for the four-form field strength are

$$d * F_{(4)} = -\frac{1}{2}F_{(4)}\wedge F_{(4)} \quad , \quad dF_{(4)} = 0. \tag{2.18}$$

We now consider the following ansatz for M-theory Gödel Universes

$$ds^2 = -\left(dt + \sum_{i=1}^5 c_i\rho_i^2 d\phi_i\right)^2 + \sum_{i=1}^5\left(d\rho_i^2 + \rho_i^2 d\phi_i^2\right), \tag{2.19}$$

$$F_{\rho_i\phi_i\rho_j\phi_j} = \rho_i\rho_j a_{ij}, \tag{2.20}$$

with $i,j = 1,...,5$, $c_i \geq 0$ and a_{ij} a symmetric constant matrix with $a_{ii} = 0$. Thus we have ten independent constants in a_{ij}. The c_i are constants that we can take to be positive without loss of generality. To obtain a $G_{2n+1} \times R^{10-2n}$ Gödel universe one can put $c_{n+1} = \cdots = c_5 = 0$ and the other c_i non-zero.

We should also note that this is not the most general ansatz one can have for supersymmetric Gödel Universes in M-theory. Indeed, in Section 2.4 we give two examples of supersymmetric solutions that does not fit into the above ansatz. However, we believe that all the M-theory Gödel Universes with maximal supersymmetry, for a given n, should fit into the above ansatz.

We first consider the Einstein equations. Define

$$b_i = \sum_j a_{ij}^2. \tag{2.21}$$

Then the non-zero components of $K^\mu{}_\nu$ are

$$K^t{}_t = -\frac{1}{12}\sum_i b_i \quad , \quad K^t{}_{\phi_i} = -\frac{1}{2}c_i \rho_i^2 b_i, \quad , \quad K^{\rho_i}{}_{\rho_i} = K^{\phi_i}{}_{\phi_i} = \frac{1}{2}b_i - \frac{1}{12}\sum_j b_j. \tag{2.22}$$

By considering all the Einstein equations we can now see that all EOMs are satisfied if

$$c_i^2 = \frac{1}{4}b_i - \frac{1}{24}\sum_j b_j, \tag{2.23}$$

or, equivalently, if

$$b_i = 4c_i^2 + 4\sum_j c_j^2. \tag{2.24}$$

Note that as consequence of equation (2.23) the c_i's are uniquely determined by giving the a_{ij} matrix, since we imposed that $c_i \geq 0$.

Consider now the two remaining equations (2.18). The Bianchi identity is trivially fulfilled. The equations of motion for the field strength can be written as

$$\partial_\sigma \left(\sqrt{-g} F^{\sigma\mu\nu\xi} \right) = \frac{1}{2}\frac{1}{(4!)^2}\epsilon^{\mu\nu\xi\kappa_1\cdots\kappa_8}\, F_{\kappa_1\cdots\kappa_4}F_{\kappa_5\cdots\kappa_8}, \tag{2.25}$$

where $\epsilon^{01\cdots\sharp} = 1$ (we denote the eleventh coordinate by $x^\sharp$). Using that $\sqrt{-g} = \rho_1 \cdots \rho_5$ we get that (2.25) is satisfied if[6]

$$-2\sum_j a_{ij}c_j = \frac{1}{4}\sum_{j,k,l,m} \eta^{ijklm} a_{jk}a_{lm}, \tag{2.26}$$

where

$$\eta^{ijklm} = \begin{cases} 1 & \text{if } i,j,k,l,m \text{ are all different} \\ 0 & \text{otherwise} \end{cases} . \tag{2.27}$$

[6]The factor of 1/4 on the RHS is present because this means that e.g. $\frac{1}{4}\sum_{j,k,l,m}\eta^{1jklm}a_{jk}a_{lm} = a_{23}a_{45} + a_{24}a_{35} + a_{25}a_{34}$.

Thus, for a given collection of c_i and a_{ij} the EOMs of M-theory reduce to the equations (2.23) and (2.26). Since (2.23) determines the c_i uniquely, we can see (2.26) as a condition that a given a_{ij} matrix should fulfil in order to correspond to a solution.

Note that if we start with the c_i's given, we can try to solve for a_{ij}. Since we have ten independent components of a_{ij} and ten independent equations (2.24) and (2.26) this looks possible. This is not the case, however, there can be many different matrices a_{ij} giving the same c_i's.

For use in the rest of the paper, we introduce here a shorthand expression for a given Gödel Universe solution

$$(c_1, c_2, c_3, c_4, c_5) \quad , \qquad (a_{12}, a_{13}, a_{14}, a_{15}, a_{23}, a_{24}, a_{25}, a_{34}, a_{35}, a_{45}), \tag{2.28}$$

that specifies uniquely an M-theory Gödel Universe background.

Supersymmetry conditions for M-theory Gödel Universes

To count the number of supersymmetries of a classical solution in M-theory, we need to count the number of independent spinors η that obey the Killing spinor equation

$$D_\mu \eta + \frac{1}{288}\left(\Gamma_\mu{}^{\nu\sigma\kappa\xi} - 8\delta_\mu^\nu \Gamma^{\sigma\kappa\xi}\right)\eta F_{\nu\sigma\kappa\xi} = 0, \tag{2.29}$$

with $D_\mu \eta = \partial_\mu \eta + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}\eta$. The Gamma-matrices Γ^μ are for the curved space, while Γ^a are for the tangent space.

For our ansatz (2.19)-(2.20) it is convinient to classify constant spinors according to their eigenvalues under the matrices

$$J_1 = \Gamma^{012} \quad , \qquad J_2 = \Gamma^{034} \quad , \qquad J_3 = \Gamma^{056} \quad , \qquad J_4 = \Gamma^{078} \quad , \qquad J_5 = \Gamma^{09\sharp}. \tag{2.30}$$

Here the Gamma matrices are for the tangent space. The eigenvalues of J_i are ± 1. Then since we impose $(\Gamma_\sharp)^2 = \Gamma_{01\cdots\sharp} = 1$ we have the constraint

$$\prod_{i=1}^{5} J_i = -1, \tag{2.31}$$

Thus, we can classify constant spinors according to their eigenvalues of (J_1, J_2, J_3, J_4). Note that any eigenvalue of (J_1, J_2, J_3, J_4) has degeneracy two.

Our ansatz for a Killing spinor obeying (2.29) is

$$\eta^{(s_1,s_2,s_3,s_4)} = \left[1 + \sum_{k=1}^{5} u_k^{(s_1,s_2,s_3,s_4)} \Gamma^0 \left(\Gamma^{2k} x^{2k-1} - \Gamma^{2k-1} x^{2k}\right)\right] \eta_0^{(s_1,s_2,s_3,s_4)}, \tag{2.32}$$

where $\eta_0^{(s_1,s_2,s_3,s_4)}$ is a constant spinor with eigenvalues (s_1, s_2, s_3, s_4) of (J_1, J_2, J_3, J_4). The Gamma matrices in this expression are flat. The Killing spinor equation (2.29) then becomes

$$\left(6\sum_{i=1} c_i J_i - \sum_{i<j} a_{ij} J_i J_j\right) \eta^{(s_1,s_2,s_3,s_4)} = 0, \tag{2.33}$$

for the time-component, and

$$\partial_{2k-1}\eta^{(s_1,s_2,s_3,s_4)} = \Gamma^0\Gamma^{2k} L\eta^{(s_1,s_2,s_3,s_4)} \quad , \quad \partial_{2k}\eta^{(s_1,s_2,s_3,s_4)} = -\Gamma^0\Gamma^{2k-1} L\eta^{(s_1,s_2,s_3,s_4)}, \tag{2.34}$$

$$L \equiv -\frac{1}{12} J_k \sum_{i<j} a_{ij} J_i J_j + \frac{1}{4}\sum_{i=1}^{5} a_{ki} J_i - \frac{1}{2} c_k, \tag{2.35}$$

for the spatial components. This reduces now to the equations

$$6\sum_{i=1} c_i s_i = \sum_{i<j} a_{ij} s_i s_j \quad , \quad L\left(\eta^{(s_1,s_2,s_3,s_4)} - \eta_0^{(s_1,s_2,s_3,s_4)}\right) = 0, \tag{2.36}$$

provided we take

$$u_k^{(s_1,s_2,s_3,s_4)} = -\frac{1}{12} s_k \sum_{i<j} a_{ij} s_i s_j + \frac{1}{4}\sum_{i=1}^{5} a_{ki} s_i - \frac{1}{2} c_k. \tag{2.37}$$

In conclusion, the number of supersymmetries for a given choice of c_i and a_{ij} is given by two times the number of solutions to (2.36) when trying the 16 possible choices of (s_1, s_2, s_3, s_4).

Symmetries of the ansatz

We can also consider the possible symmetries of the ansatz (2.19)-(2.20) for a given Gödel Universe solution. I.e. for given c_i and a_{ij}, what general transformations can give $\tilde{c}_i$ and $\tilde{a}_{ij}$ that also solves the EOMs (2.23) and (2.26), and have the same amount of supersymmetry (which is computed from (2.36)). We have three basic transformations that map solutions to solutions keeping the same amount of supersymmetry.

The first one is that we can rescale the c_i and a_{ij}, i.e. the transformation is $\tilde{a}_{ij} = \gamma a_{ij}$ and $\tilde{c}_i = \gamma c_i$ with $\gamma \neq 0$.

The second one is that we interchange two of the five different two-planes. If we for example exchange the 12-plane and 34-plane we should make the transformation $\tilde{a}_{12} = a_{12}$, $\tilde{a}_{i1} = a_{i2}$, $\tilde{a}_{i2} = a_{i1}$ and $\tilde{c}_1 = c_2$, $\tilde{c}_2 = c_1$, $\tilde{c}_i = c_i$ with $i, j = 3, 4, 5$.

The third transformation is that we can change parity in one of the planes. If we for example change parity in the 9♮-plane we should make the transformation $\tilde{a}_{ij} = -a_{ij}$, $\tilde{a}_{i5} = a_{i5}$ and $\tilde{c}_i = c_i$, $\tilde{c}_5 = -c_5$ with $i, j = 1, 2, 3, 4$.

2.4 Solutions with $n = 2$ and $n = 4$

$n = 2$ Gödel Universes

In [7] an $n = 2$ Gödel Universe background of M-theory with 20 supersymmetries was found. In [10] this was shown to be T-dual to a pp-wave with 24 supersymmetries. We review this briefly and put the T-duality into a broader picture involving also the S-duality transformation of Section 2.2.

In [16] the Penrose limit of $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$ from intersecting D3-branes was found to be the type IIB pp-wave background with the metric

$$ds^2 = -2dx^+dx^- - \beta^2 \sum_{i=1}^{4} (\tilde{x}^i)^2 (dx^+)^2 + \sum_{i=1}^{4} (d\tilde{x}^i)^2 + \sum_{i=5}^{8} (dx^i)^2, \tag{2.38}$$

and the RR fields

$$F^{(5)} = -2\beta dx^+ (d\tilde{x}^1 d\tilde{x}^2 + d\tilde{x}^3 d\tilde{x}^4)(dx^5 dx^6 + dx^7 dx^8). \tag{2.39}$$

We see the metric corresponds to $a_1 = a_2 = 1$ in Section 2.1. After the T-duality transformation of Section 2.1 we get the Gödel Universe background of type IIA

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{4} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{8} (dx^i)^2 + dy^2, \tag{2.40}$$

$$H_{12y} = H_{34y} = -2\beta \quad , \qquad F_{1256} = F_{1278} = F_{3456} = F_{3478} = -2\beta, \tag{2.41}$$

with $J_{12} = J_{34} = 1$. Lifting to M-theory we get the $G_5 \times \mathbb{R}^6$ Gödel Universe background

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{4} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{\sharp} (dx^i)^2, \tag{2.42}$$

$$F_{1256} = F_{1278} = F_{3456} = F_{3478} = F_{129\sharp} = F_{349\sharp} = -2\beta, \tag{2.43}$$

where $x^9 = y$. This M-theory Gödel Universe background has 20 supersymmetries [7].[7] In our notation of Section 2.3 we can write this solution as

$$(1, 1, 0, 0, 0) \quad , \qquad (0, -2, -2, -2, -2, -2, -2, 0, 0, 0). \tag{2.44}$$

If we T-dualize the pp-wave background (2.38)-(2.39) in the x^7 and x^8 directions we have the same metric but the RR fields are given by

$$F_{(3)} = 2\beta dx^+ (d\tilde{x}^1 d\tilde{x}^2 + d\tilde{x}^3 d\tilde{x}^4). \tag{2.45}$$

This type IIB pp-wave with 24 supersymmetries can be obtained from a Penrose limit of $\mathrm{AdS}_3 \times S^3 \times \mathbb{R}^4$ from the D1-D5 system [6, 17, 18, 19, 20]. If we do the T-duality of Section 2.1 we get the Gödel Universe metric (2.40) and the NS-NS and RR fields

$$H_{12y} = H_{34y} = -2\beta \quad , \qquad F_{12} = F_{34} = -2\beta \quad , \qquad F_{012y} = F_{034y} = 2\beta, \tag{2.46}$$

This Gödel Universe background of type IIA has also 20 supersymmetries. Indeed this Gödel Universe background is related by T-dualities in the x^7 and x^8 directions to the background (2.40)-(2.41).

[7] Notice that the number of supersymmetries can be changed under T-duality as we examine in Appendix B.

We now uplift the type IIA Gödel Universe (2.40), (2.46) to M-theory. We call $y = x^9$ and the eleventh direction u. We then get

$$ds^2 = -dt^2 + du^2 + \sum_{i=1}^{9} (dx^i)^2 - 2\beta \sum_{i,j=1}^{4} J_{ij} x^i dx^j (dt + du), \tag{2.47}$$

$$F_{0129} = F_{u129} = F_{0349} = F_{u349} = 2\beta. \tag{2.48}$$

This is an M-theory pp-wave solution with 24 supersymmetries, obtained from a Penrose limit in [18]. We see that we have connected the type IIA Gödel Universe (2.40), (2.46) to the M-theory pp-wave (2.47)-(2.48) by the S-duality transformation described in Section 2.2.

$$\begin{tikzcd}
& \text{IIA Gödel Universe} \arrow[dl, leftrightarrow, "\mathrm{T}(x^9)"'] \arrow[dr, leftrightarrow, "\mathrm{S}(u)"] & \\
\text{IIB D1-D5 pp-wave} \arrow[d, leftrightarrow, "\mathrm{S}"] & & \text{M-theory pp-wave} \arrow[d, leftrightarrow, "\mathrm{S}(x^9)"] \\
\text{IIB F1-NS5 pp-wave} \arrow[rr, leftrightarrow, "\mathrm{T}(x^8)"] & & \text{IIA F1-NS5 pp-wave}
\end{tikzcd}$$

Figure 1: Diagram that shows connection between T and S-duality transformations relating Gödel Universes and pp-waves.

We can now demonstrate that the T-duality transformation of Section 2.1 and the S-duality transformation of Section 2.2 in fact are equivalent when we have at least two flat directions, i.e. when $n \leq 3$ for the Gödel Universes. Consider the M-theory pp-wave (2.47)-(2.48). Compactify now this on T^3 in the x^8, $y = x^9$ and the u directions. We have then illustrated in Figure 1 a duality chain that connects the T-duality of Section 2.1 and the S-duality of Section 2.2. Starting from the M-theory pp-wave (2.47)-(2.48) we can S-dualize along $y = x^9$ to obtain the type IIA pp-wave which comes from a Penrose limit of F1-NS5 with 24 supersymmetries [17, 21]. This we can T-dualize along x^8 to obtain the F1-NS5 pp-wave in type IIB. Using type IIB S-duality we arrive at the D1-D5 pp-wave (2.38), (2.45). Then we do the T-duality of Section 2.1 along $y = x^9$ to obtain the type IIA Gödel Universe (2.40), (2.46). Finally, we end up with the M-theory pp-wave (2.47)-(2.48) by uplift to M-theory. In conclusion we have shown that for $n \leq 3$ the T and S-duality that relate pp-waves to Gödel Universes are equivalent for M-theory on T^3.

$n = 2$ Gödel Universe/pp-wave mixture

It is interesting to consider the mixed pp-wave background of type IIB with metric (2.38) and NS-NS and RR three form fluxes [6]

$$F_{(3)} = 2\beta \cos\gamma \, dx^+ (d\tilde{x}^1 d\tilde{x}^2 + d\tilde{x}^3 d\tilde{x}^4) \quad , \qquad H_{(3)} = -2\beta \sin\gamma \, dx^+ (d\tilde{x}^1 d\tilde{x}^2 + d\tilde{x}^3 d\tilde{x}^4). \tag{2.49}$$

This background has 24 supersymmetries. We now T-dualize this according to Section 2.1 with $a_1 = a_2 = 1$. We subsequently do two trivial T-dualities along x^8 and x^7, and then make a trivial uplift to M-theory. The resulting M-theory background is

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{4} J_{ij} x^i dx^j \right)^2 + \left(dy - \beta \sin\gamma \sum_{i,j=1}^{4} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{9} (dx^i)^2, \tag{2.50}$$

$$\begin{aligned} F_{0129} = F_{0349} = -2\beta \sin\gamma \quad , \qquad F_{y129} = F_{y349} = -2\beta, \\ F_{1256} = F_{1278} = F_{3456} = F_{3478} = -2\beta \cos\gamma, \end{aligned} \tag{2.51}$$

where $J_{12} = J_{34} = 1$ and x^9 is the eleventh direction.

This solution is interesting for several reasons. If we consider the special case $\gamma = 0$ it reduces to the M-theory $n = 2$ Gödel Universe (2.42)-(2.43). On the other hand, if we consider the case $\gamma = \pi/2$ it becomes an M-theory pp-wave with 24 supersymmetries [18]. The solution (2.50)-(2.51) thus interpolates between being a Gödel Universe and a pp-wave.

Moreover, we see that for $\gamma = \pi/2$ we have an example of a supersymmetric pp-wave solution with NS-NS flux that does not become a Gödel Universe background after the T-duality of Section 2.1.

In general the solution (2.50)-(2.51) has 16 supersymmetries while, as we just saw, it has 20 supersymmetries when $\gamma = 0$ and 24 supersymmetries when $\gamma = \pi/2$. We give a proof of the 16 supersymmetries of the solution in Appendix B.

Note that the solution (2.50)-(2.51) does not fit into the ansatz (2.19)-(2.20) since the g_{yi} and the F_{0ijk} components are non-zero. This is of course in accordance with the fact that the solution (2.50)-(2.51) can describe both a pp-wave and a Gödel Universe.

To understand better the interpolation between Gödel Universe and pp-wave we write the metric (2.50) in polar coordinates

$$\begin{aligned} ds^2 \, &= -dt^2 + dy^2 + \sum_{i=5}^{9} (dx^i)^2 + \sum_{i=1}^{2} \left[d\rho_i^2 + \rho_i^2 \left(1 - \beta^2 \cos^2\gamma \, \rho_i^2 \right) d\phi_i^2 \right] \\ &\quad -2\beta^2 \cos^2\gamma \, \rho_1^2 \rho_2^2 d\phi_1 d\phi_2 - 2\beta \left(dt + \sin\gamma \, dy \right) \left(\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2 \right). \end{aligned} \tag{2.52}$$

If we consider for example the 12-plane, we see that for the curve $\rho_1 =$ constant it is a closed space-like curve for $\rho_1 < 1/(\beta \cos\gamma)$ and a closed time-like curve for $\rho_1 > 1/(\beta\cos\gamma)$. If we now consider the limit $\cos\gamma \to 0$ we see that the necessary radius to make a closed time-like curves goes to infinity. So if $\cos\gamma$ is very small but non-zero the geometry is almost that of a pp-wave and the radii of the closed time-like curves are so large that they effectively can be ignored. It would be interesting to study how the holographic sheets behave in this limit.

$n = 4$ **Gödel Universes**

In [10] it was pointed out that the maximally supersymmetric pp-wave of type IIB [4] can be T-dualized into an $n = 4$ Gödel Universe solution. In the following we examine this T-duality carefully and study its supersymmetry. We find that the T-duality can give two inequivalent $n = 4$ Gödel Universes. We then go on to discuss a more general type of pp-wave solution and the consequences this solution have for the relation between the T and S-dualities of Sections 2.1 and 2.2. In particular we find a new Gödel Universe with 20 supersymmetries.

Consider the maximally symmetric pp-wave of type IIB [4] with metric

$$ds^2 = -2dx^+dx^- - \beta^2 \sum_{i=1}^{8} (\tilde{x}^i)^2 (dx^+)^2 + \sum_{i=1}^{8} (d\tilde{x}^i)^2, \tag{2.53}$$

and RR flux

$$F_{(5)} = 4\beta dx^+ \left(d\tilde{x}^1 d\tilde{x}^2 d\tilde{x}^3 d\tilde{x}^4 + d\tilde{x}^5 d\tilde{x}^6 d\tilde{x}^7 d\tilde{x}^8 \right). \tag{2.54}$$

We see that the metric corresponds to $a_1^2 = a_2^2 = a_3^2 = a_4^2 = 1$ in Section 2.1. However, we shall see in the following that the signs of the a_i's become important. Without loss of generality we therefore choose $a_1 = a_2 = a_3 = 1$ and $a_4 = s$, where $s = \pm 1$.

We consider now the T-duality of Section 2.1. We get the metric

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{8} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{8} (dx^i)^2 + dy^2, \tag{2.55}$$

with $J_{12} = J_{34} = J_{56} = 1$ and $J_{78} = s$. The NS-NS and RR fluxes are

$$H_{12y} = H_{34y} = H_{56y} = -2\beta \quad , \quad H_{78y} = -2s\beta \quad , \quad F_{1234} = F_{5678} = 4\beta. \tag{2.56}$$

As we shall see below, this defines two inequivalent $n = 4$ type IIA Gödel Universe backgrounds, corresponding to $s = \pm 1$.

We can now uplift these solutions to M-theory. This gives the $n = 4$ M-theory Gödel Universes

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{8} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{\sharp} (dx^i)^2, \tag{2.57}$$

$$F_{129\sharp} = F_{349\sharp} = F_{569\sharp} = -2\beta \quad , \quad F_{789\sharp} = -2s\beta \quad , \quad F_{1234} = F_{5678} = 4\beta, \tag{2.58}$$

with $y = x^9$ as usual. In our short hand notation of Section 2.3 we can write the solutions as

$$(1,1,1,s,0) \quad , \quad (4,0,0,-2,0,0,-2,4,-2,-2s). \tag{2.59}$$

Using the methods in Section 2.3 it is now easy to compute that the number of supersymmetries is zero if $s = 1$ and 20 if $s = -1$. For the detail of Killing spinors see (B.9) in the Appendix B. This is obviously also true for the type IIA Gödel Universes (2.55)-(2.56). We also computed the 20 supersymmetries of the $s = -1$ type IIA solution in an alternative way in Appendix B. We denote the $s = -1$ M-theory Gödel Universe as $G_9 \times R^2$.

If we want to write down the $s = -1$ solution with $c_4 = 1$ we note that using the parity changing transformation of Section 2.3 we get the equivalent solution

$$(1,1,1,1,0) \quad , \quad (-4,0,0,2,0,0,2,4,2,2), \tag{2.60}$$

with 20 supersymmetries.

We now go on to study a one-parameter type IIB pp-wave solution with 28 supersymmetries and its dual type IIA and M-theory Gödel Universe. As we shall see, this illustrates that the T and S dualities in Section 2.1 and 2.2 are not equivalent for $n = 4$. It also has other interesting features.

The type IIB pp-wave solution with 28 supersymmetries [20] (see also [22]) has metric

$$ds^2 = -2dx^+dx^- - \beta^2 \sum_{i=1}^{8} (\tilde{x}^i)^2 (dx^+)^2 + \sum_{i=1}^{8} (d\tilde{x}^i)^2, \tag{2.61}$$

and RR fluxes

$$\begin{aligned} F_{(3)} &= 2\beta \sin\gamma dx^+ \left(d\tilde{x}^1 d\tilde{x}^2 + d\tilde{x}^3 d\tilde{x}^4 + d\tilde{x}^5 d\tilde{x}^6 - d\tilde{x}^7 d\tilde{x}^8 \right), \\ F_{(5)} &= 4\beta \cos\gamma dx^+ \left(d\tilde{x}^1 d\tilde{x}^2 d\tilde{x}^3 d\tilde{x}^4 + d\tilde{x}^5 d\tilde{x}^6 d\tilde{x}^7 d\tilde{x}^8 \right), \end{aligned} \tag{2.62}$$

γ is the parameter of the solution. When $\gamma = 0$ we regain the maximally supersymmetric pp-wave of type IIB [4]. When $\gamma = \pi/2$ we get a type IIB pp-wave with 28 supersymmetries found in [20, 22].

We now take the T-duality in Section 2.1. From the $\gamma = 0$ case above we know that to get a supersymmetric solution we should take $a_1 = a_2 = a_3 = -a_4 = 1$. The T-duality then gives the type IIA Gödel Universe background

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{8} J_{ij} x^i dx^j \right)^2 + \sum_{i=1}^{8} (dx^i)^2 + dy^2, \tag{2.63}$$

with $J_{12} = J_{34} = J_{56} = 1$ and $J_{78} = -1$. The NS-NS and RR fluxes are

$$\begin{aligned} F_{12} = F_{34} = F_{56} = -F_{78} = -2\beta\sin\gamma \quad &, \quad F_{012y} = F_{034y} = F_{056y} = -F_{078y} = 2\beta\sin\gamma, \\ H_{12y} = H_{34y} = H_{56y} = -H_{78y} = -2\beta \quad &, \quad F_{1234} = F_{5678} = 4\beta\cos\gamma. \end{aligned} \tag{2.64}$$

This type IIA Gödel Universe solution has generically 12 supersymmetries, (see Appendix B for a proof of this). In the two special cases $\gamma = 0$ and $\gamma = \pi/2$ it has instead 20 supersymmetries (see Appendix B for the $\gamma = \pi/2$ case). Therefore we see that we have two different $n = 4$ Gödel Universes of type IIA with 20 supersymmetries which are not related by dualities.

We now uplift the solution (2.63)-(2.64) to M-theory. We get the M-theory background with metric

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{8} J_{ij} x^i dx^j\right)^2 + \left(du - \beta \sin\gamma \sum_{i,j=1}^{8} J_{ij} x^i dx^j\right)^2 + \sum_{i=1}^{9} (dx^i)^2. \qquad (2.65)$$

and four-form fluxes

$$F_{0129} = F_{0349} = F_{0569} = -F_{0789} = 2\beta \sin\gamma \quad , \qquad F_{1234} = F_{5678} = 4\beta \cos\gamma,$$
$$F_{u129} = F_{u349} = F_{u569} = -F_{u789} = 2\beta. \qquad (2.66)$$

where we denote $y = x^9$ and u is the eleventh direction. This solution has many common features with the $n = 2$ Gödel Universe/pp-wave mixture (2.50)-(2.51). For $\gamma = 0$ it reduces to the M-theory Gödel Universe (2.57)-(2.58). For $\gamma = \pi/2$ it becomes an M-theory pp-wave with 22 supersymmetries which was found in [23]. Thus, the solution interpolates between a Gödel Universe and a pp-wave, just as (2.50)-(2.51) does. The structure of the metrics and fluxes are clearly also of the same form.

The solution (2.65)-(2.66) has in general 12 supersymmetries while it clearly has 20 supersymmetries when $\gamma = 0$ and 22 supersymmetries when $\gamma = \pi/2$ (see Appendix B for a proof of this).

Finally we note that if we set $\gamma = \pi/2$ we can illustrate that we do not have any equivalence between the T and S-dualities of Sections 2.1 and 2.2, in the sense of the duality chain depicted in Figure 1, for $n = 4$. For $\gamma = \pi/2$ (2.61)-(2.62) describes a type IIB pp-wave with 28 supersymmetries. The T-duality of Section 2.1 gives the type IIA Gödel Universe (2.63)-(2.64) with 20 supersymmetries. Uplifting this to M-theory by the S-duality of Section 2.2 gives an M-theory pp-wave with 22 supersymmetries. Thus, clearly the two pp-wave solutions related to the type IIA Gödel Universe cannot be related via any trivial dualities since they do not have the same amount of supersymmetry.

2.5 Solutions with $n = 1$ and $n = 3$

$n = 1$ Gödel Universes

To find an $n = 1$ Gödel Universe, we start with the type IIB pp-wave solution

$$ds^2 = -2dx^+ dx^- - \beta^2 \sum_{i=1}^{2} (\tilde{x}^i)^2 (dx^+)^2 + \sum_{i=1}^{2} (d\tilde{x}^i)^2 + \sum_{i=3}^{8} (dx^i)^2, \qquad (2.67)$$

$$F^{(3)} = 2\beta dx^+ d\tilde{x}^1 d\tilde{x}^2, \qquad (2.68)$$

which is the Penrose limit of a D5-brane. This pp-wave has 16 supersymmetries and is S-dual to the Nappi-Witten model [24, 17], which is the Penrose limit of an NS5-brane [25].

We then do the T-duality of Section 2.1 with $a_1 = 1$. This gives the type IIA Gödel Universe

$$ds^2 = -\Big[dt + \beta(x^1dx^2 - x^2dx^1)\Big]^2 + \sum_{i=1}^{8}(dx^i)^2 + dy^2, \tag{2.69}$$

$$H_{12y} = -2\beta \quad , \quad F_{12} = -2\beta \quad , \quad F_{012y} = 2\beta. \tag{2.70}$$

Clearly this becomes a pp-wave when uplifted to M-theory, it in fact becomes a pp-wave which is directly related to the pp-wave of (2.67)-(2.68). If we instead T-dualize the above type IIA solution along x^3 and x^4 then we can trivially uplift this to M-theory and we get the M-theory Gödel Universe

$$ds^2 = -\left[dt + \beta(x^1dx^2 - x^2dx^1)\right]^2 + \sum_{i=1}^{\sharp}(dx^i)^2 \quad , \quad F_{1234} = F_{5678} = -F_{129\sharp} = 2\beta, \tag{2.71}$$

where we put $y = x^9$ and $x^\sharp$ is the eleventh direction. The $G_3 \times \mathbb{R}^8$ Gödel Universe (2.71) has 8 supersymmetries. We can write the solution as

$$(1,0,0,0,0) \quad , \quad (2,0,0,-2,0,0,0,2,0,0), \tag{2.72}$$

in the notation of Section 2.3.

$n = 3$ Gödel Universes

Using the general formulas in [23] we can construct the following M-theory pp-wave background with 20 supersymmetries

$$ds^2 = -2dx^+dx^- - \beta^2\left(4\Big[(\tilde{x}^1)^2 + (\tilde{x}^2)^2\Big] + \sum_{i=3}^{6}(\tilde{x}^i)^2\right)(dx^+)^2 + \sum_{i=1}^{6}(d\tilde{x}^i)^2 + \sum_{i=7}^{9}(dx^i)^2, \tag{2.73}$$

$$F_{(4)} = \beta dx^+\left(4d\tilde{x}^1d\tilde{x}^2dx^7 + 2d\tilde{x}^3d\tilde{x}^4dx^7 + 2d\tilde{x}^5d\tilde{x}^6dx^7\right). \tag{2.74}$$

If we perform an S-duality along x^9 and the T-duality in Section 2.1 with $a_1 = 2$ and $a_2 = a_3 = 1$ we get the type IIB Gödel Universe

$$ds^2 = -\left(dt + \beta\sum_{i,j=1}^{6} J_{ij}x^idx^j\right)^2 + \sum_{i=1}^{8}(dx^i)^2 + dy^2, \tag{2.75}$$

$$\begin{gathered} H_{12y} = -4\beta \quad , \quad H_{34y} = H_{56y} = -2\beta \quad , \quad F_{127} = -4\beta \quad , \quad F_{347} = F_{567} = -2\beta, \\ F_{0127y} = -F_{34568} = 4\beta \quad , \quad F_{0347y} = F_{0567y} = -F_{12568} = -F_{12348} = 2\beta, \end{gathered} \tag{2.76}$$

with $J_{12} = 2$ and $J_{34} = J_{56} = 1$. We now do a T-duality along x^8 and then we uplift the solution to M-theory. This gives the M-theory background

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{6} J_{ij} x^i dx^j\right)^2 + \sum_{i=1}^{\sharp} (dx^i)^2, \tag{2.77}$$

$$\begin{gathered} F_{3456} = F_{1278} = F_{129\sharp} = -4\beta \\ F_{1256} = F_{1234} = F_{3478} = F_{5678} = F_{349\sharp} = F_{569\sharp} = -2\beta, \end{gathered} \tag{2.78}$$

with $y = x^9$ and $x^\sharp$ the eleventh direction. This $G_7 \times \mathbb{R}^4$ Gödel universe in M-theory preserves 14 supersymmetries. We can write the solution as

$$(2,1,1,0,0) \quad , \quad (-2,-2,-4,-4,-4,-2,-2,-2,-2,0). \tag{2.79}$$

2.6 Solutions with $n = 5$

We consider here M-theory Gödel Universes with $n = 5$. These backgrounds cannot be related to pp-waves in the same way as for the $n \leq 4$ Gödel Universes. To find these backgrounds we have therefore instead used the methods of Section 2.3. The basic method used was to search through the possible c_i's with $c_i, a_{ij} \in \mathbb{Z}$. However, using these results as input we have been able to find an $n = 5$ solution with 18 supersymmetries and moreover a quite general family of solutions with at least 16 supersymmetries.

Consider the following Gödel Universe background

$$(2,1,1,1,1) \quad , \quad (-6,2,2,2,0,0,0,4,4,4), \tag{2.80}$$

This is an $n = 5$ Gödel Universe with 18 supersymmetries. We denote it G_{11}. We believe that this is the only possible $n = 5$ Gödel Universe with 18 supersymmetries, up to symmetries of the ansatz, and that it is not possible to find an $n = 5$ Gödel Universe with more supersymmetry. The metric and four-form flux are

$$ds^2 = -\left(dt + \beta \sum_{i,j=1}^{\sharp} J_{ij} x^i dx^j\right)^2 + \sum_{i=1}^{\sharp} (dx^i)^2, \tag{2.81}$$

$$F_{1234} = -6\beta \quad , \quad F_{1256} = F_{1278} = F_{129\sharp} = 2\beta \quad , \quad F_{5678} = F_{569\sharp} = F_{78\sharp} = 4\beta, \tag{2.82}$$

with $J_{12} = 2$ and $J_{34} = J_{56} = J_{78} = J_{9\sharp} = 1$. In Appendix B we give the explicit expressions for the Killing spinors of this background.

The G_{11} solution (2.80) is actually part of a family of solutions

$$(k+1,k,k,k,1) \quad , \quad (-4k-2,2,2,2k,0,0,2k-2,4k,2k+2,2k+2). \tag{2.83}$$

This solution preserves at least 16 supersymmetries for any $k \in \mathbb{R}$. We see that it reduces to (2.80) for $k = 1$. Using the symmetries of the ansatz of Section 2.3 we can in fact transform (2.83) into the following general solution

$$(p, p, p, p - q, sq) \ , \ (-4sp, 0, 2sq, 2p - 2q, 0, 2sq, 2p - 2q, 4sp - 2sq, 2p + 2q, 2p), \qquad (2.84)$$

with $p, q \in \mathbb{R}$ and $s = \pm 1$. The general solution (2.84) preserves at least 16 supersymmetries. In fact all the $n = 5$ solutions with at least 16 supersymmetries that we have found can fit into (2.84). It is therefore natural to conjecture that any M-theory $n = 5$ Gödel Universe with at least 16 supersymmetries is a special case of (2.84). We do not have a proof of this at present.

We can also consider the special values of p and q which give more than 16 supersymmetries for (2.84). For $p = q = 0$ we have $n = 0$ and 32 supersymmetries, for $p = q$ or $p = 0$ we have $n = 2$ and 20 supersymmetries, for $q = 0$ we have $n = 4$ and 20 supersymmetries and finally for $q = 2p$ or $p = -q$ we have $n = 5$ and 18 supersymmetries.

However, not all solutions of the ansatz (2.19)-(2.20) which have at least 16 supersymmetries are special cases of (2.84). A counter-example is the $n = 4$ Gödel Universe

$$(2, 2, 1, 1, 0) \quad , \qquad (0, -6, 2, 4, -2, 6, 4, 0, 2, 2), \qquad (2.85)$$

which preserves 16 supersymmetries.

We have also found several $n = 5$ Gödel Universe solutions with less than 16 supersymmetries. We have studied all solutions of the ansatz (2.19)-(2.20) with $c_i, a_{ij} \in \mathbb{Z}$ and $1 \le c_i \le 5$. Of these we have found one solution with 18 supersymmetries, being (2.80), several solutions with 16 supersymmetries all of which are special cases of (2.84), several solutions with 12 and 14 supersymmetries and one solution without supersymmetry. Altogether we found 43 solutions. We show here a list of five of these solutions.

14 susy	$(3, 3, 2, 1, 1)$	$(0, -10, 4, 4, -2, 8, 8, 2, 2, 4)$,
12 susy	$(4, 1, 1, 1, 1)$	$(-6, -6, -6, -6, -4, -4, -4, -4, -4, -4)$,
12 susy	$(5, 2, 1, 1, 1)$	$(-6, -8, -8, -8, -6, -6, -6, -4, -4, -4)$,
14 susy	$(5, 5, 4, 2, 2)$	$(0, -18, 6, 6, -2, 14, 14, 4, 4, 8)$.
no susy	$(5, 5, 4, 2, 2)$	$(-16, 6, -10, 2, 6, 2, -10, -12, -12, 8)$.

3 String Theory on Gödel Universes

In this section we compute the string spectrum for two Gödel Universe background of type IIA string theory. Our method is to consider the corresponding type IIB pp-wave background, compactified on a circle of radius R, and quantize the string theory in the light-cone gauge. Then we take the limit $R \to 0$ and obtain the spectrum in the type IIA Gödel Universe. We consider mainly the $n = 4$ Gödel Universe (2.55)-(2.56) T-dual to the maximally supersymmetric pp-wave (2.53)-(2.54), but also the $n = 2$ Gödel Universe (2.40)-(2.41) T-dual to the pp-wave (2.38)-(2.39) is briefly considered.

3.1 String Spectrum on compactified pp-wave

In this section we compute the string spectrum in the compactified maximally supersymmetric type IIB pp-wave background (2.53)-(2.54) in the coordinate system of (2.3). Since we are only interested in the supersymmetric case we choose $s = -1$.

We consider first the bosonic fields in the world-sheet theory alone. The light-cone gauge is defined by (see e.g. [26, 17])

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau + 2wR\sigma \quad (0 \le \sigma \le \pi), \tag{3.1}$$

where w is the winding number along the compactified circle. The momentum is quantized as follows

$$p^+ = E + \frac{m}{R}, \quad p^- = \frac{E}{2} - \frac{m}{2R} \quad (m \in \mathbf{Z}). \tag{3.2}$$

with $p^+ = -p_-$ and $p^- = -p_+$. The world-sheet action in the light-cone gauge is given by

$$S = \frac{1}{\pi\alpha'} \int d\tau d\sigma \left[\partial_+ X^i \partial_- X^i - \beta\alpha' p^+ J_{ij} X^i \partial_\tau X^j + \beta w R J_{ij} X^i \partial_\sigma X^j \right], \tag{3.3}$$

where we defined $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$. It is also useful to employ the complex fields

$$Z^1 = X^1 + iX^2, \quad Z^2 = X^3 + iX^4, \quad Z^3 = X^5 + iX^6, \quad Z^4 = X^7 - iX^8. \tag{3.4}$$

The EOMs is given by

$$(\partial_\tau^2 - \partial_\sigma^2) Z^a = 4i\beta\alpha' p^+ \partial_\tau Z^a - 4i\beta w R \partial_\sigma Z^a. \quad (a = 1, 2, 3, 4) \tag{3.5}$$

If we perform the field redefinition

$$Z^a = Z_0^a e^{2i\beta p^+ \tau + 2i\beta^a w R\sigma} (= Z_0^a e^{i\beta X^+}), \tag{3.6}$$

then we can write the EOMs as

$$(\partial_\tau^2 - \partial_\sigma^2) Z_0^a + 4f^2 Z_0^a = 0, \tag{3.7}$$

where we defined

$$f = \sqrt{(\beta\alpha' p^+)^2 - (\beta w R)^2}. \tag{3.8}$$

The boundary condition for Z_0^a is given by

$$Z_0^a(\tau, \sigma + \pi) = e^{-2i\beta w R\pi} Z_0^a(\tau, \sigma). \tag{3.9}$$

The fields $Z_0^a(\tau, \sigma)$ thus obey the same EOMs as in the usual maximally supersymmetric IIB pp-wave. However, it is important to notice that they obey the twisted boundary condition (3.9). The mode expansion is given by ($\delta = \beta w R$)

$$Z_0^a = i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z}} \left(\frac{\alpha_{n+\delta}^a}{\omega_n^+} e^{-2i\omega_n^+ \tau - 2i(n+\delta)\sigma} + \frac{\tilde{\alpha}_{n-\delta}^a}{\omega_n^-} e^{-2i\omega_n^- \tau + 2i(n-\delta)\sigma} \right), \tag{3.10}$$

$$\bar{Z}_0^a = i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z}} \left(\frac{\bar{\alpha}_{n-\delta}^a}{\omega_n^-} e^{-2i\omega_n^- \tau - 2i(n-\delta)\sigma} + \frac{\bar{\tilde{\alpha}}_{n+\delta}^a}{\omega_n^+} e^{-2i\omega_n^+ \tau + 2i(n+\delta)\sigma} \right), \tag{3.11}$$

where we defined

$$\omega_n^+ = \begin{cases} \sqrt{(n+\delta)^2+f^2} & \text{for } n \geq -\delta \\ -\sqrt{(n+\delta)^2+f^2} & \text{for } n < -\delta \end{cases} \quad , \tag{3.12}$$

$$\omega_n^- = \begin{cases} \sqrt{(n-\delta)^2+f^2} & \text{for } n > \delta \\ -\sqrt{(n-\delta)^2+f^2} & \text{for } n \leq \delta \end{cases} \quad . \tag{3.13}$$

We consider now the light-cone gauge quantization of the string theory. The quantization of the oscillators is again the same as in the maximally supersymmetric IIB pp-wave with respect to Z_0^a fields,

$$[\alpha^a_{n+\delta}, \bar{\alpha}^b_{m-\delta}] = 2\omega_n^+ \delta_{n+m,0}\delta_{a,b} \quad , \quad [\tilde{\alpha}^a_{n-\delta}, \bar{\tilde{\alpha}}^b_{m+\delta}] = 2\omega_n^- \delta_{n+m,0}\delta_{a,b} \quad . \tag{3.14}$$

We now compute the spectrum of the string theory by imposing the Virasoro constraints $T_{++} = T_{--} = 0$. Before we computing the spectrum, let us note that

$$p^- = -\frac{\delta S}{\delta \dot{X}^+} = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left[2\dot{X}^- + \beta J_{ij} X^i \dot{X}^j \right] \quad . \tag{3.15}$$

From this we can find $X^- = 2\alpha' p^- \tau - wR\sigma + \cdots$. The condition $T_{++} + T_{--} = 0$ leads to

$$E^2 - \left(\frac{m}{R}\right)^2 - \left(\frac{wR}{\alpha'}\right)^2 = \frac{2}{\alpha'} \sum_{n\in\mathbb{Z}} \left(N_n^+ \sqrt{(n+\delta)^2+f^2} + N_n^- \sqrt{(n-\delta)^2+f^2} \right) + 2\beta p^+ \mathbf{J} \ , \tag{3.16}$$

with the angular momentum operator

$$\mathbf{J} = \sum_{n\in\mathbb{Z}} \left(N_n^+ - N_n^- \right) \ , \tag{3.17}$$

and the number operators

$$N_n^+ = \begin{cases} \sum_{a=1}^4 \frac{1}{2\omega_n^+} \tilde{\alpha}^a_{-n-\delta} \bar{\tilde{\alpha}}^a_{n+\delta} & \text{for } n \geq -\delta \\ \sum_{a=1}^4 \frac{1}{2\omega_{-n}^-} \alpha^a_{n+\delta} \bar{\alpha}^a_{-n-\delta} & \text{for } n < -\delta \end{cases} \quad , \tag{3.18}$$

$$N_n^- = \begin{cases} \sum_{a=1}^4 \frac{1}{2\omega_n^-} \bar{\tilde{\alpha}}^a_{-n+\delta} \tilde{\alpha}^a_{n-\delta} & \text{for } n > \delta \\ \sum_{a=1}^4 \frac{1}{2\omega_{-n}^+} \bar{\alpha}^a_{n-\delta} \alpha^a_{-n+\delta} & \text{for } n \leq \delta \end{cases} \quad . \tag{3.19}$$

In the above we ignored the zero-point energy since it is cancelled by the fermionic one in the supersymmetric case $s = -1$.

The other constraint $T_{++} - T_{--} = 0$ leads to the level matching

$$\sum_{n=-\infty}^{\infty} n(N_n^+ + N_n^-) + mw = 0. \tag{3.20}$$

Finally, we would like to note that the spectrum (3.16) can also equivalently be found from the light-cone Hamiltonian

$$
\begin{aligned}
H &= \frac{1}{8(\alpha')^2 p^+}\int_0^\pi d\sigma\left[\dot{Z}^a\dot{\bar{Z}}^a + Z^{a'}\bar{Z}^{a'} - 2i\beta wR(Z^a\bar{Z}^{a'} - \bar{Z}^a Z^{a'})\right],\\
&= \frac{1}{8(\alpha')^2 p^+}\int_0^\pi d\sigma\left[\dot{Z}^a_0\dot{\bar{Z}}^a_0 + Z^{a'}_0\bar{Z}^{a'}_0 + 2i\beta\alpha' p^+(Z^a_0\dot{\bar{Z}}^a_0 - \bar{Z}^a_0\dot{Z}^a_0) + 4((\beta\alpha' p^+)^2 - (\beta Rw)^2)\right].
\end{aligned}
$$

Note that $H = p^-$ with p^- given by (3.15) after imposing the Virasoro constraints.

We now turn to the contribution to the spectrum from world-sheet fermions. We start with the general action $S_F = \frac{1}{\pi\alpha'}\int d\tau d\sigma \mathcal{L}_F$ with [27]

$$
\mathcal{L}_F = i(\eta^{ab}\delta_{IJ} - \epsilon^{ab}\rho_{3IJ})\partial_a X^\mu \bar{\theta}^I\Gamma_\mu D_b\theta^J, \tag{3.21}
$$

where θ^1 and θ^2 are Green-Schwarz fermions, each of which is a Majorana-Weyl spinor in ten dimensions (sixteen components), and where $\rho_3 = \sigma_3$. The covariant derivative is given by[8]

$$
D_a = \partial_a + \frac{1}{4}\partial_a X^\alpha\left(\omega_{\mu\nu\alpha}\Gamma^{\mu\nu} - \frac{1}{4\cdot 5!}F_{\mu\nu\rho\lambda\kappa}\Gamma^{\mu\nu\rho\lambda\kappa}(i\sigma_2)\Gamma_\alpha\right), \tag{3.22}
$$

where the non-trivial components of the spin connection are given by $\omega_{+ij} = \beta J_{ij}$. After we impose the light-cone gauge $\Gamma^+\theta^{1,2} = 0$ for fermions, we find the following Lagrangian for the fermions[9]

$$
\begin{aligned}
\mathcal{L}_F = &\ 4i(\alpha' p^+ - wR)\bar{\theta}^1\Gamma_+\left(\partial_+ - i\beta(\alpha' p^+ + wR)J\right)\theta^1\\
&+4i(\alpha' p^+ + wR)\bar{\theta}^2\Gamma_+\left(\partial_- - i\beta(\alpha' p^+ - wR)J\right)\theta^2\\
&-8i\beta\left((\alpha' p^+)^2 - (wR)^2\right)\bar{\theta}^1\Gamma^+\Gamma^{1234}\theta^2,
\end{aligned} \tag{3.23}
$$

where J denotes the spin $J = \frac{i}{4}J_{ij}\Gamma^{ij}$ of the field $\theta^{1,2}$. After normalizing the fermions (both have eight components as spinors in 8 dimensions) such that $\theta^1 = 2\sqrt{\alpha' p^+ - wR}\ S^1$ and $\theta^2 = 2\sqrt{\alpha' p^+ + wR}\ S^2$, we obtain

$$
\begin{aligned}
\mathcal{L}_F = i\Big(&S^1\left(\partial_+ - i\beta(\alpha' p^+ + wR)J\right)S^1 + S^2\left(\partial_- - i\beta(\alpha' p^+ - wR)J\right)S^2\\
&-2\beta\sqrt{(\alpha' p^+)^2 - (wR)^2}\,S^1\gamma^{1234}S^2\Big),
\end{aligned} \tag{3.24}
$$

where we define the Gamma-matrices in eight dimensions by γ^i $(i = 1, 2, \cdots, 8)$. Defining the fermionic fields $\mathcal{S}^1$ and $\mathcal{S}^2$ as follows

$$
\begin{aligned}
S^1(\tau,\sigma) &= \exp\left(2i\beta J(\alpha' p^+\tau + wR\sigma)\right)\mathcal{S}^1(\tau,\sigma),\\
S^2(\tau,\sigma) &= \exp\left(2i\beta J(\alpha' p^+\tau - wR\sigma)\right)\mathcal{S}^2(\tau,\sigma),
\end{aligned} \tag{3.25}
$$

[8] Notice that the gamma matrix Γ^μ is of the curved spacetime not of the local Minkowski frame. However, in the discussion below we can neglect this difference as one can see explicitly.

[9] To derive this we use the properties of Γ matrices $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$, $(\Gamma^0)^T = -\Gamma^0$, $(\Gamma^i)^T = \Gamma^i$. It turns out that only the component $\partial_a X^+$ contributes in (3.21) and (3.22).

then the action is equivalent to that of the maximally supersymmetric IIB pp-wave. Notice that, as for the bosonic fields, we have a twisted boundary condition

$$\mathcal{S}^{1,2}(\tau,\sigma+\pi) = e^{-2i\beta\pi J w R}\mathcal{S}^{1,2}(\tau,\sigma). \tag{3.26}$$

To diagonalize the matrix J we consider the eigenvalues of $(i\Gamma_{12}, i\Gamma_{34}, i\Gamma_{56}, i\Gamma_{78})$ since the spin J is given by $J = \frac{i}{2}(\Gamma_{12}+\Gamma_{34}+\Gamma_{56}-\Gamma_{78})$. Then the eight possible cases can be divided into four with $J=1$, i.e. $(+,+,+,+), (-,+,+,-), (+,-,+,-), (+,+,-,-)$, and four with $J=-1$, i.e. $(-,+,-,+), (-,-,+,+), (+,-,-,+), (-,-,-,-)$. Here we used the constraint $\gamma^{1234}S^{1,2} = \gamma^{5678}S^{1,2}$. In this way we have found that the fermionic fields $\mathcal{S}^1$ and $\mathcal{S}^2$ have the same twisted boundary condition (3.26) as the bosonic ones. This is due to supersymmetry and leads to vanishing zero-point energy. Since complex conjugation of a spinor change the sign of its eigenvalue under J, we can conveniently use the complex fermion fields S^{Ii} with the charge $J=-1$ and $\bar{S}^{Ii}$ $(I=1,2,\ i=1,2,3,4)$ with $J=1$ below.

We can now compute the fermionic part of the light-cone Hamiltonian

$$\begin{aligned} H &= \frac{i}{32\pi(\alpha')^2 p^+}\int_0^\pi d\sigma\left[\bar{S}^1\dot{S}^1 + S^1\dot{\bar{S}}^1 + \bar{S}^2\dot{S}^2 + S^2\dot{\bar{S}}^2\right], \\ &= \frac{i}{32\pi(\alpha')^2 p^+}\int_0^\pi d\sigma\left[\bar{\mathcal{S}}^1\dot{\mathcal{S}}^1 + \mathcal{S}^1\dot{\bar{\mathcal{S}}}^1 + \bar{\mathcal{S}}^2\dot{\mathcal{S}}^2 + \mathcal{S}^2\dot{\bar{\mathcal{S}}}^2 + 2i\beta\alpha' p^+ J(\bar{\mathcal{S}}^1\mathcal{S}^1 + \mathcal{S}^2\bar{\mathcal{S}}^2)\right]. \end{aligned} \tag{3.27}$$

We expand the (complex) fermionic field as follows $(i=1,2,3,4)$

$$\begin{aligned} S^{1i}(\sigma,\tau) &= \sqrt{\alpha'}\sum_{n\in\mathbb{Z}}\left[\frac{f}{\sqrt{f^2+(\omega_n^+-f)^2}}e^{-2i\omega_n^+\tau+2i(n+\delta)\sigma}S^i_{n+\delta}\right. \\ &\quad \left. +\frac{i(\omega_n^- -(n-\delta))}{\sqrt{f^2+(\omega_n^- -f)^2}}e^{-2i\omega_n^-\tau-2i(n-\delta)\sigma}(\Pi\tilde{S}^i_{n-\delta})\right], \\ S^{2i}(\sigma,\tau) &= \sqrt{\alpha'}\sum_{n\in\mathbb{Z}}\left[\frac{f}{\sqrt{f^2+(\omega_n^- -f)^2}}e^{-2i\omega_n^-\tau-2i(n-\delta)\sigma}\tilde{S}^i_{n-\delta}\right. \\ &\quad \left. -\frac{i(\omega_n^+ -(n+\delta))}{\sqrt{f^2+(\omega_n^+ -f)^2}}e^{-2i\omega_n^+\tau+2i(n+\delta)\sigma}(\Pi S^i_{n+\delta})\right], \end{aligned} \tag{3.28}$$

where $\Pi = \gamma^{1234}$ and $\delta = \beta w R$ as above. The quantization of oscillators is given by

$$\{S^i_{n+\delta}, \bar{S}^j_{m-\delta}\} = 2\delta_{n+m,0}\delta_{ij} \quad , \quad \{\tilde{S}^i_{n-\delta}, \bar{\tilde{S}}^j_{m+\delta}\} = 2\delta_{n+m,0}\delta_{ij}\ . \tag{3.29}$$

Finally we can write the Hamiltonian (3.27) as follows

$$H = \frac{1}{\alpha' p^+}\sum_{n\in\mathbb{Z}}\left(F_n^+\sqrt{(n+\delta)^2+f^2} + F_n^-\sqrt{(n-\delta)^2+f^2}\right) + \beta\sum_{n\in\mathbb{Z}}(F_n^+ - F_n^-)\ , \tag{3.30}$$

with the fermionic number operators given by

$$F_n^+ = \begin{cases} \frac{1}{2}\sum_{i=1}^4 \tilde{S}^i_{-n-\delta}\bar{\tilde{S}}^i_{n+\delta} & \text{for } n \geq -\delta \\ \frac{1}{2}\sum_{i=1}^4 S^i_{n+\delta}\bar{S}^i_{-n-\delta} & \text{for } n < -\delta \end{cases}\ , \tag{3.31}$$

$$F_n^- = \begin{cases} \frac{1}{2}\sum_{i=1}^4 \tilde{\bar{S}}^i_{-n+\delta}\tilde{S}^i_{n-\delta} & \text{for } n > \delta \\ \frac{1}{2}\sum_{i=1}^4 \bar{S}^i_{n-\delta}S^i_{-n+\delta} & \text{for } n \leq \delta \end{cases} . \tag{3.32}$$

From (3.16) and (3.30) we see that the total spectrum of the string theory on the compactified maximally supersymmetric pp-wave background (2.53)-(2.54) is

$$\begin{aligned} E^2 - \left(\frac{m}{R}\right)^2 - \left(\frac{wR}{\alpha'}\right)^2 &= 2\beta p^+ \sum_{n\in\mathbb{Z}} \left(N_n^+ + F_n^+ - N_n^- - F_n^-\right) \\ +\frac{2}{\alpha'}\sum_{n\in\mathbb{Z}} \Big((N_n^+ + F_n^+)&\sqrt{(n+\delta)^2+f^2} + (N_n^- + F_n^-)\sqrt{(n-\delta)^2+f^2}\Big) , \end{aligned} \tag{3.33}$$

with level matching condition

$$\sum_{n\in\mathbb{Z}} n(N_n^+ + N_n^- + F_n^+ + F_n^-) + mw = 0. \tag{3.34}$$

We see that the supersymmetry is manifest in these expressions.

3.2 String Spectrum on Gödel Universe

We consider now the T-dual background in type IIA string theory and take the limit $R \to 0$. Then we find that the string spectrum on the type IIA Gödel Universe with 20 supersymmetries (2.55)-(2.56) is given by

$$\begin{aligned} E^2 - p_y^2 = \frac{2}{\alpha'}\sum_{n\in\mathbb{Z}} \Big[&(N_n^+ + F_n^+)\sqrt{(n+\alpha'\beta p_y)^2 + \beta^2\alpha'^2(E^2-p_y^2)} \\ +(N_n^- + F_n^-)&\sqrt{(n-\alpha'\beta p_y)^2 + \beta^2\alpha'^2(E^2-p_y^2)}\Big] + 2\beta E\sum_{n\in\mathbb{Z}}\left(N_n^+ + F_n^+ - N_n^- - F_n^-\right) , \end{aligned} \tag{3.35}$$

along with the level matching condition

$$\sum_{n\in\mathbb{Z}} n(N_n^+ + N_n^- + F_n^+ + F_n^-) = 0. \tag{3.36}$$

As a check of (3.35) we compare now the spectrum of the bosonic zero-modes to the spectrum computed from supergravity. If we restrict to the zero-modes, then (3.35) becomes

$$E^2 - p_y^2 = 2\beta E(\mathbf{N} + \mathbf{J} + \epsilon_0), \tag{3.37}$$

where we set $N_0^+ + N_0^- + F_0^+ + F_0^- = \mathbf{N} + \epsilon_0$ and $\epsilon_0(\geq 0)$ represents the fermionic zero-point energy (see e.g. [27] for the detailed analysis of the zero-point energy in the pp-wave).

Let us for example consider the dilaton field φ ($\epsilon_0 = 4$) in the Gödel background and compare the above result with that obtained from the low energy supergravity analysis. The EOM is given by the massless Klein-Gordon equation

$$\Box\varphi = \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}\right)\varphi + \sum_{a=1}^4\left(\triangle_a + \beta^2\rho_a^2\frac{\partial^2}{\partial t^2} - 2\beta\frac{\partial^2}{\partial t\partial\phi_a}\right)\varphi = 0, \tag{3.38}$$

where we denote the Laplacian by $\triangle_a = \frac{1}{\rho_a}\frac{\partial}{\partial\rho_a}(\rho_a\frac{\partial}{\partial\rho_a}) + \frac{1}{\rho_a^2}\frac{\partial^2}{\partial\phi_{a2}}$. We assume the form

$$\varphi = e^{-iEt+i\mathbf{J_a}\phi_a+ip_y y} f(\rho), \tag{3.39}$$

and then we obtain the harmonic system

$$\sum_{a=1}^{4}\left(-\triangle_a + (\beta E)^2\rho_a^2\right) f(\rho) = \left(E^2 - p_y^2 - 2\beta E\mathbf{J}\right) f(\rho), \tag{3.40}$$

where $\mathbf{J} = \sum_{a=1}^{4}\mathbf{J}_a$. By using standard results of harmonic oscillators we find the spectrum

$$E^2 - p_y^2 = 2\beta E(\mathbf{N} + \mathbf{J} + 4), \tag{3.41}$$

where $\mathbf{N} = n_1 + n_2 + \cdots + n_8$ is the familiar quantum number of eight harmonic oscillators. This result exactly matches with the string theory result (3.37).

Clearly, the equation for the spectrum (3.35) is of a rather intricate nature since it has energy both on the left- and right-hand side. In order to gain a better understanding of (3.35) we consider the special case where $p_y = 0$. We then get

$$\frac{E^2}{2\beta} = \sum_{n\in\mathbb{Z}}\sqrt{E^2 + \left(\frac{n}{\beta\alpha'}\right)^2}\left(\mathcal{N}_n^+ + \mathcal{N}_n^-\right) + E\sum_{n\in\mathbb{Z}}\left(\mathcal{N}_n^+ - \mathcal{N}_n^-\right) \ , \tag{3.42}$$

where we define

$$\mathcal{N}_n^+ = N_n^+ + F_n^+ \ , \qquad \mathcal{N}_n^- = N_n^- + F_n^- \ . \tag{3.43}$$

If we consider the case where the string only has low-lying string modes $|n| \ll \beta\alpha' E$ the spectrum can be written

$$E^3 = 4\beta E^2\sum_{n\in\mathbb{Z}}\mathcal{N}_n^+ + \frac{1}{\beta(\alpha')^2}\sum_{n\in\mathbb{Z}} n^2\left(\mathcal{N}_n^+ + \mathcal{N}_n^-\right) \tag{3.44}$$

If we can further neglect the first term, for example by considering $\mathcal{N}_n^+ = 0$, we see that we get a very unconventional string spectrum.

If we instead consider the case where the string only has high excitations $|n| \gg \beta\alpha' E$ the spectrum can be written as

$$E^2 = \frac{2}{\alpha'}\sum_{n\in\mathbb{Z}}|n|\left(\mathcal{N}_n^+ + \mathcal{N}_n^-\right) + 2\beta E\sum_{n\in\mathbb{Z}}\left(\mathcal{N}_n^+ - \mathcal{N}_n^-\right) + \beta^2\alpha' E^2\sum_{n\in\mathbb{Z}}\frac{1}{|n|}\left(\mathcal{N}_n^+ + \mathcal{N}_n^-\right) \ , \tag{3.45}$$

where the last term is small compared to the first term. If we throw away the last term we can write

$$E = \beta\sum_{n\in\mathbb{Z}}\left(\mathcal{N}_n^+ - \mathcal{N}_n^-\right) + \sqrt{\left[\beta\sum_{n\in\mathbb{Z}}\left(\mathcal{N}_n^+ - \mathcal{N}_n^-\right)\right]^2 + \frac{2}{\alpha'}\sum_{n\in\mathbb{Z}}|n|\left(\mathcal{N}_n^+ + \mathcal{N}_n^-\right)} \tag{3.46}$$

We see that we regain the flat-space string spectrum if we can neglect the angular momentum part.

Finally, going back to the spectrum (3.35) with p_y, we note that if we take the $\beta^2\alpha' \to \infty$ limit we get the spectrum

$$E = \frac{1}{2}p_y + \sqrt{\frac{1}{4}p_y^2 + 4\beta \sum_{n\in\mathbb{Z}} \mathcal{N}_n^+} \tag{3.47}$$

The limit $\beta^2\alpha' \to \infty$ means the geometry become strongly curved as we can understand from (2.15). As we can see from (3.47) this means the masses of the string states become negligible. This is why the spectrum (3.47) effectively becomes that of the supergravity modes (3.37). This limit is obviously interesting since the radii of the closed time-like curves would be able to become arbitrarily small in this limit if the supergravity description was valid. We also note that the spectrum (3.47) suggest that one can make a string bit model similar to the one of [28, 29, 30, 31] on the maximally supersymmetric pp-wave.

3.3 $n = 2$ Gödel Universe in type IIA

As the next case we examine the string theory in the Gödel background which is T-dual to the supersymmetric pp-wave defined by (2.38) and (2.39). This background is given by (2.40) and (2.41) and preserves 20 supersymmetries as we have seen above.

The computation of spectrum in the compactified pp-wave can be performed as in the previous case. For example, the fermionic part is given by

$$\begin{aligned}\mathcal{L}_F = i\Big(&S^1\left(\partial_+ - i\beta(\alpha' p^+ + wR)J\right)S^1 + S^2\left(\partial_- - i\beta(\alpha' p^+ - wR)J\right)S^2 \\ &-\beta\sqrt{(\alpha' p^+)^2 - (wR)^2}\,S^1(\Gamma^{1256} + \Gamma^{3456})S^2\Big),\end{aligned} \tag{3.48}$$

where the matrix J is given by $J = \frac{i}{4}\sum_{i,j=1}^4 J_{ij}\Gamma^{ij}$. It is useful to take the following linear combinations for each of the two eight components spinors S_a^1 and S_a^2 $(a = 1, 2, \cdots, 8)$

$$\begin{aligned}&S_1 : (+,+,+,+),\quad S_2 : (+,+,-,-),\quad S_3 : (+,-,+,-),\quad S_4 : (-,+,+,-),\\ &\bar{S}_1 : (-,-,-,-),\quad \bar{S}_2 : (-,-,+,+),\quad \bar{S}_3 : (-,+,-,+),\quad \bar{S}_4 : (+,-,-,+),\end{aligned} \tag{3.49}$$

where we specified the eigenvalue ± 1 of $(i\Gamma_{12}, i\Gamma_{34}, i\Gamma_{56}, i\Gamma_{78})$. Then it is easy to see that

$$(\Gamma^{12} + \Gamma^{34})S_{3,4} = (\Gamma^{12} + \Gamma^{34})\bar{S}_{3,4} = 0, \tag{3.50}$$

and thus these four spinors out of eight are massless, while the other four are massive. The value of J is given by zero for $S_{3,4}, \bar{S}_{3,4}$ and $+1$ (-1) for $S_{1,2}$ $(\bar{S}_{1,2})$. This charge distribution (or equally the twisted boundary condition) is the same as in the bosonic fields in the world-sheet theory and this is again due to the supersymmetry in the compactified solution.

The string spectrum is given by the previous formula (3.16) with respect to the $Z^{1,2}, S_{1,2}$ $(J = 1)$ and $\bar{Z}^{1,2}, \bar{S}_{1,2}$ $(J = -1)$, while we should set $\beta = 0$ with respect to $Z^{3,4}, S_{3,4}$

and $\bar{Z}^{3,4}, \bar{S}_{3,4}$ $(J = 0)$ excitations. The zero-energy does vanish due to the remaining supersymmetry. The small radius limit leads to the Gödel Universe model and the spectrum is given by

$$E^2 - p_y^2 = \frac{2}{\alpha'} \sum_{n\in\mathbb{Z}} \left[(N_n^+ + F_n^+) \sqrt{(n + \alpha'\beta p_y)^2 + \beta^2\alpha'^2(E^2 - p_y^2)} \right.$$
$$\left. + (N_n^- + F_n^-) \sqrt{(n - \alpha'\beta p_y)^2 + \beta^2\alpha'^2(E^2 - p_y^2)} \right] + 2\beta E \sum_{n\in\mathbb{Z}} \left(N_n^+ + F_n^+ - N_n^- - F_n^- \right)$$
$$+ \sum_{i=1}^{4} p_i^2 + \frac{2}{\alpha'} \sum_{n=-\infty}^{\infty} |n|(N_n + F_n) \ , \tag{3.51}$$

along with the level matching condition

$$\sum_{n\in\mathbb{Z}} n(N_n^+ + N_n^- + N_n + F_n^+ + F_n^- + F_n) = 0. \tag{3.52}$$

where $N_n^{\pm}, F_n^{\pm}$ and N_n, F_n counts the number of oscillators with spin $J = \pm 1$ and $J = 0$, respectively, and p_i are the momenta in the four extra transverse directions. Obviously, other supersymmetric Gödel Universe backgrounds of string theory can be treated similarly using these methods.

4 D-branes on Gödel Universes

In this section we consider the D-brane spectrum on Gödel Universes which are T-dual to highly supersymmetric pp-waves. We obtain the D-brane spectrum by taking the T-duality transformation of that in the pp-wave. Below we investigate this issue from the viewpoint of both the boundary condition in the world-sheet theory and the classical brane solutions in supergravity. We consider the cases of the Gödel Universes (2.55)-(2.56) and (2.40), (2.46).

4.1 Boundary Conditions in String Theory

We discuss here the D-branes in the type IIA $n = 4$ Gödel Universe with 20 supersymmetries, defined by (2.55) and (2.56) with $s = -1$, by applying T-duality to the known D-brane spectrum in the maximally supersymmetric type IIB pp-wave. We consider here the D-branes on the pp-wave that have Neumann boundary conditions in the x^+ and x^- directions. Then the half-BPS D-brane in the pp-wave background is given by the Dp-branes ($p = 3, 5, 7$) which have the boundary condition $(+, -, 2, 0)$, $(+, -, 3, 1)$ and $(+, -, 4, 2)$ respectively[10] [33] (see also [34, 32]). The spectrum also includes the 1/4 BPS D1-brane $(+, -, 0, 0)$ [32]. Below we study the T-dual counterparts of these branes.

To investigate the D-brane spectrum we consider in the following the boundary conditions for D-branes (see also [35] for similar analysis in the exactly solvable model of magnetic

[10] Here the symbol $(+, -, a, b)$ means Neumann boundary conditions for $x^+, x^-, \tilde{x}^{i_1}, ..., \tilde{x}^{i_a}, \tilde{x}^{j_1}, ..., \tilde{x}^{j_b}$ $(1 \leq i_1, \cdots, i_a \leq 4, \quad 5 \leq j_1, \cdots, j_a \leq 8)$ following the convention [32].

universe [26]). For simplicity we discuss only the bosonic part of the boundary conditions. Under the T-duality in the $y = \frac{1}{2}x^+ - x^-$ direction the world-sheet fields are transformed as follows.[11]

$$
\begin{aligned}
\partial_D y &= \partial_N \tilde{y} + \beta(\rho_1^2 \partial_D \phi_1 + \rho_2^2 \partial_D \phi_2 + \rho_3^2 \partial_D \phi_3 - \rho_4^2 \partial_D \phi_4), \\
\partial_N y &= \partial_D \tilde{y} + \beta(\rho_1^2 \partial_N \phi_1 + \rho_2^2 \partial_N \phi_2 + \rho_3^2 \partial_N \phi_3 - \rho_4^2 \partial_N \phi_4),
\end{aligned} \tag{4.2}
$$

where we defined $\partial_N = \partial + \bar{\partial} = \partial_\sigma$ and $\partial_D = \bar{\partial} - \partial = -i\partial_\tau$. Here and in the rest of this section we call the T-duality direction y in the pp-wave background and $\tilde{y}$ in the T-dual Gödel Universe background.

In order to take the T-duality we assume the D-brane is placed away from the origin at fixed $x^i \neq 0 \quad (i = 1, 2, \cdots, 8)$. Thus, while the D-brane is at a fixed position in the x^i coordinates (and therefore in the ρ_k and ϕ_k coordinates), the position of the D-brane is rotating as $\tilde{x}^{2k-1} = \rho_k \cos(\phi_k - a_k \beta x^+)$ and $\tilde{x}^{2k} = \rho_k \sin(\phi_k - a_k \beta x^+) \quad (k = 1, 2, 3, 4)$ in the original coordinates of the pp-wave (2.53). These shifted D-brane configurations can be obtained by a symmetry transformation described in [36] and they have the same amount of supersymmetry as before.

D0-branes from D1-branes

Before the T-duality a D1-brane on the pp-wave has the following boundary conditions

$$\partial_D \rho_i = 0, \quad \partial_D \phi_i = 0, \quad (i = 1, 2, 3, 4) \tag{4.3}$$

$$\partial_N x^+ = 0, \tag{4.4}$$

$$\partial_N x^- + \beta(\rho_1^2 \partial_N \phi_1 + \rho_2^2 \partial_N \phi_2 + \rho_3^2 \partial_N \phi_3 - \rho_4^2 \partial_N \phi_4) = 0, \tag{4.5}$$

where (4.5) comes from the consideration of symmetries acting on boundary conditions [36]. After rewriting the above boundary conditions using (4.2), we get

$$\partial_N t + \beta(\rho_1^2 \partial_N \phi_1 + \rho_2^2 \partial_N \phi_2 + \rho_3^2 \partial_N \phi_3 - \rho_4^2 \partial_N \phi_4) = 0, \tag{4.6}$$

$$\partial_D \tilde{y} = 0, \tag{4.7}$$

with (4.3). Finally let us compare this result with the general boundary conditions[12]

$$G_{\mu\nu} \partial_N X^\nu + (B_{\mu\nu} + F_{\mu\nu}) \partial_D X^\nu |_{\partial\Sigma} = 0 \quad \text{(mixed Neumann)} \tag{4.8}$$

$$\text{or} \quad \partial_D f(X^\mu)|_{\partial\Sigma} = 0 \quad \text{(Dirichlet)}. \tag{4.9}$$

In conclusion the boundary conditions (4.6) and (4.7) shows that the D-brane obtained by T-duality is a D0-brane in the Gödel background. Indeed its world-volume $x^i = y = const.$ is equivalent to the time-like geodesic line.

[11] For a general formula see e.g. [26] It is given by

$$
\begin{aligned}
\partial \tilde{y} &= -B_{y\mu} \partial x^\mu - g_{yy}(\partial y + G_{y\mu} \partial x^\mu), \\
\bar{\partial} \tilde{y} &= -B_{y\mu} \bar{\partial} x^\mu + g_{yy}(\bar{\partial} y + G_{y\mu} \bar{\partial} x^\mu).
\end{aligned} \tag{4.1}
$$

[12] Here the index μ in the mixed Neumann condition should be taken along the world-volume direction.

D2 and D4-branes from D3-branes

Let us consider a D3-brane whose world-volume is along $(x^+, x^-, \tilde{x}^1, \tilde{x}^2)$ in the pp-wave background. The computations are almost the same as above. We obtain (4.7) again by T-duality (the other conditions are the same as before T-duality). Thus we get a D2-brane extended in the x^1, x^2 which means we have a "small" Gödel Universe in the world-volume directions.

A more interesting case is the D3-brane whose world-volume is given by $(x^+, x^-, \tilde{x}^1, \tilde{x}^3)$. In this case we start with the following boundary conditions in the pp-wave

$$\partial_N \rho_{1,2} = 0, \quad \partial_D \rho_{3,4} = 0, \tag{4.10}$$
$$\partial_D(\phi_1 - \beta x^+) = \partial_D(\phi_2 - \beta x^+) = 0, \quad \partial_D \phi_{3,4} = 0, \tag{4.11}$$
$$\partial_N x^+ = 0, \tag{4.12}$$
$$\partial_N x^- + \beta(\rho_3^2 \partial_N \phi_3 - \rho_4^2 \partial_N \phi_4) = 0. \tag{4.13}$$

After we take T-duality, the result is given by (4.6) and

$$\partial_D(\phi_1 - \phi_2) = 0, \tag{4.14}$$
$$\beta(\rho_1^2 \partial_N \phi_1 + \rho_2^2 \partial_N \phi_2) + \partial_D \tilde{y} = 0, \tag{4.15}$$
$$\partial_N \tilde{y} + \partial_D t + \beta(\rho_1^2 \partial_D \phi_1 + \rho_2^2 \partial_D \phi_2) - \frac{1}{\beta} \partial_D \phi_1 = 0. \tag{4.16}$$

Comparing with (4.8) and (4.9), we find that the resulting system represents a D4-brane whose world-volume coordinate is given by $t, \tilde{y}, \rho_1, \rho_2$ and $\theta = (\phi_1 + \phi_2)/2$ with the gauge flux

$$F_{t\tilde{y}} = -1, \quad F_{\tilde{y}\theta} = -\frac{1}{\beta}. \tag{4.17}$$

D6-branes from D5-branes

We can assume that the world-volume of D5-brane is in the direction of $(x^+, x^-, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^5)$. Then we find almost the same boundary conditions as for the D4-brane from the D3-brane, and the result is given by a D6-brane extending in the direction $t, \tilde{y}, \rho_1, \phi_1, \rho_2, \rho_3$ and $\theta = (\phi_2 + \phi_3)/2$ with the flux (4.17).

D6 and D8-branes from D7-branes

If we consider the D7-brane extending in the direction $(x^+, x^-, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4, \tilde{x}^5, \tilde{x}^6)$, then we get D6-brane (with no gauge flux) in the Gödel model as in the same way as in the D1-brane in pp-wave.

For the D7-brane whose world-volume is given by $(x^+, x^-, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4, \tilde{x}^5, \tilde{x}^7)$, we obtain a D8-brane in the direction $t, \tilde{y}, \rho_1, \rho_2, \rho_3, \rho_4, \phi_1, \phi_2$ and $\theta = (\phi_3 + \phi_4)/2$ with the gauge flux (4.17) in the same way as when we got a D4-brane from a D3-brane.

Summary of D-branes in the Gödel model

In this way we get D0,D2,D4,D6 and D8-branes in the Gödel background. In particular we have obtained the D-branes with both electric and magnetic fluxes for D4,D6 and D8-branes. Apart from the D0-branes, the above list of D-branes have world-volumes which include closed time-like curves so that we have "small" Gödel Universes along their world-volumes. It would be interesting to study the gauge theories on these world-volumes.

4.2 Classical Brane Solutions in Supergravity

We now consider D-branes on Gödel Universes from the viewpoint of the supergravity solutions. The example we discuss here is that of a D4-brane in the $n = 2$ Gödel model given by (2.40), (2.46) with 20 supersymmetries. We find the supergravity solution for a smeared D4-brane by performing T-duality transformations of a D5-brane solution in the pp-wave (2.38), (2.45). The D5-brane solution was found in [37] (see also [38] and [39] for related systems). Afterwards we consider the localized D4-brane.

The D5-brane solution on the pp-wave is

$$\begin{aligned} ds^2 &= h^{-\frac{1}{2}}\Big(-2dx^+dx^- - \beta^2\sum_{i=1}^{4}(\tilde{x}^i)^2(dx^+)^2 + \sum_{i=1}^{4}(d\tilde{x}^i)^2\Big) + h^{\frac{1}{2}}\sum_{j=5}^{8}(dx^j)^2, \\ h &= 1 + \frac{Ng_s\alpha'}{r^2}, \quad e^{2\phi} = g^{-1}, \quad F_{+12} = F_{+34} = 2\beta, \\ & F_{ijk} = -\epsilon_{ijkl}\partial_l h, \ (5 \leq i,j,k,l \leq 8), \end{aligned} \tag{4.18}$$

where N is the number of D5-branes and also we defined $r = \sqrt{\sum_{i=5}^{8}(\tilde{x}^i)^2}$. If we simply take the T-duality as before we obtain the smeared D4-brane solution in the Gödel model. To localize this solution we have to replace the function h with

$$f = 1 + \frac{\pi N g_s(\alpha')^{\frac{3}{2}}}{r^3} \quad , \quad r \equiv \sqrt{\sum_{i=5}^{9}(x^i)^2}, \tag{4.19}$$

as follows

$$\begin{aligned} ds^2 &= f^{-\frac{1}{2}}\Big(-(dt + \beta\sum_{i=1}^{4}J_{ij}x^i dx^j)^2 + \sum_{i=1}^{4}(dx^i)^2\Big) + f^{\frac{1}{2}}\sum_{i=5}^{9}(dx^i)^2, \\ e^{2\phi} &= f^{-\frac{1}{2}}, \quad F_{12} = F_{34} = -2\beta, \quad H_{129} = H_{349} = -2\beta, \\ F_{0129} &= F_{0349} = 2\beta, \quad F_{ijkl} = -\epsilon_{ijklm}\partial_m f, \ (5 \leq i,j,k,l,m \leq 9), \end{aligned} \tag{4.20}$$

where $y = x^9$ and $J_{12} = J_{34} = 1$. We have checked all components of the EOMs and found that this is indeed a supergravity solution.[13]

[13] To be exact we cannot express the solution by using the field strengths $F^{(4)}, H^{(3)}, F^{(2)}$ in a gauge invariant way. This is because the action include the term $|\tilde{F^{(4)}}|^2$ with $\tilde{F}^{(4)} = F^{(4)} + A^{(1)}H^{(3)}$ and the gauge transformations of the potentials are given by $\delta A^{(1)} = d\lambda$, $\delta A^{(3)} = -\lambda H^{(3)}$. Thus we should specify the value of gauge field $A^{(1)}$ and in this paper we have set $A^{(1)} = -\beta\sum_{i=1}^{4} J_{ij}x^i dx^j$.

We can also consider the M-theory lift of this solution. As discussed before we know that the lift of the Gödel Universe itself leads to the pp-wave in M-theory which preserves 24 supersymmetries (2.47) and (2.48). After changing the coordinates according to (2.2) we obtain the (localized) M5-brane solution in the pp-wave (the smeared solution can be found in [38])

$$\begin{aligned} ds^2 &= f^{-\frac{1}{3}}\Big(-2dx^+dx^- - \beta^2\sum_{i=1}^{4}(\tilde{x}^i)^2(dx^+)^2 + \sum_{i=1}^{4}(d\tilde{x}^i)^2\Big) + f^{\frac{2}{3}}\sum_{i=5}^{9}(dx^i)^2, \\ F_{+129} &= F_{+349} = 2\beta, \qquad F_{ijkl} = -\epsilon_{ijklm}\partial_m f \quad (5 \leq i,j,k,l \leq 9). \end{aligned} \tag{4.21}$$

It would be interesting to construct other Dp-brane solutions in this Gödel Universe or in other backgrounds (see [40] for classical D-brane solutions in the maximally supersymmetric pp-wave).

5 Discussion and Conclusions

In this paper we have found several new supersymmetric Gödel Universe backgrounds of string and M-theory. We discovered that not only T-duality, but also the type-IIA/M-theory S-duality can give supersymmetric Gödel Universes from pp-waves. We explained that the S-duality could be considered equivalent to the T-duality when $n \leq 3$ by considering M-theory on a three-torus. We found that there exist three inequivalent Gödel Universes with 20 supersymmetries, one of which demonstrated the fact that the S-duality transformation is not always equivalent to the T-duality.

We have found an intriguing M-theory $n = 5$ Gödel Universe with 18 supersymmetries. This background does not seem to be related to pp-waves by dualities. It would be interesting to consider if one can compactify the background to a type IIA string theory solution, perhaps this can reveal connections to other supersymmetric backgrounds. In connection with the $n = 5$ background with 18 supersymmetries we found that it is a member of a family of M-theory Gödel Universes with at least 16 supersymmetries of which the $n = 2$ and the $n = 4$ M-theory Gödel Universe with 20 supersymmetries also are members.

We found two new supersymmetric backgrounds which are mixtures of Gödel Universes and pp-waves. Perhaps one can use these backgrounds to construct a holographic dual to the string theory since we can consider a small deformation away from the pp-wave and try to deform the gauge theory dual of the pp-wave [6] so that it is dual to the deformed background.

We have also considered the string theory on two of the type IIA Gödel Universes with 20 supersymmetries. We have shown that the string theory on these backgrounds are solvable and found the string spectrum. This was done using the fact that one can light-cone quantize the T-dual pp-wave when compactified in the T-duality direction. We note that part of the spectrum looks very much like that found in [41, 42] where a space-like compactification of the maximally symmetric type IIB pp-wave was considered. Perhaps one can find a similar Penrose limit of $\mathrm{AdS}_5 \times S^5$ as in [42] giving directly the maximally supersymmetric type

IIB pp-wave (2.53)-(2.54) in the coordinate system of (2.3). This would enable one to find a dual gauge theory description of the Gödel Universe background.

We have also considered the spectrum of D-branes on the two backgrounds in which we quantized the string theory. Using again the T-dual pp-wave backgrounds we found the spectrum of D-branes on one of the backgrounds, and found a supergravity solution for a D4-brane on the other Gödel Universe background.

In this paper we have not directly addressed the perhaps most important physical question for string theory on a background of the Gödel Universe type, namely whether string theory is consistent on these backgrounds or not. Can string theory work when we have closed time-like curves?

We have seen several indications in this paper which suggest that the supersymmetric Gödel Universes are consistent string theory backgrounds. The high amount of supersymmetry for some of these backgrounds alone suggests that the backgrounds should be well-behaved. In particular they should not have closed string tachyons and should therefore be stable as string theory backgrounds. Moreover, the fact that string theory on some of the Gödel Universe backgrounds is highly solvable, i.e. that we can 1st-quantize the string theory and obtain the string spectrum without problems, also suggests that the string theory is well-defined. Finally, the fascinating fact that there are type IIA Gödel Universes which are S-dual to M-theory pp-waves means physically that string theory on those type IIA Gödel backgrounds is well-behaved at strong coupling.

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A T and S-duality conventions

T-duality conventions

Consider a metric $g_{\mu\nu}$ which has a coordinate y for which $g_{yy} = 1$. Consider moreover the NSNS 2-form potential $B_{\mu\nu}$. A T-duality in the y-direction is given by

$$\tilde{g}_{yy} = 1 \quad , \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} - g_{\mu y} g_{\nu y} + B_{\mu y} B_{\nu y} \quad , \quad \tilde{g}_{\mu y} = B_{\mu y} \tag{A.1}$$

$$\tilde{B}_{\mu\nu} = B_{\mu\nu} - B_{\mu y} g_{\nu y} + g_{\mu y} B_{\nu y} \quad , \quad \tilde{B}_{\mu y} = g_{\mu y} \tag{A.2}$$

Here $\mu, \nu \neq y$ and $\tilde{g}_{\rho\sigma}$ and $\tilde{B}_{\rho\sigma}$ are the T-dual fields. Note that a constant dilaton is mapped to a constant dilaton, i.e. $\tilde{\phi} = \phi$. If y is on a circle of radius R before the T-duality then it is on a circle of radius α'/R after the T-duality. For RR field strengths the T-duality transformations from IIA to IIB are

$$\tilde{F}_{\mu\nu\sigma\xi y} = F_{\mu\nu\sigma\xi} \quad , \quad \tilde{F}_{\mu\nu\sigma\xi\kappa} = F_{\mu\nu\sigma\xi\kappa y} \quad , \quad \tilde{F}_{\mu\nu y} = -F_{\mu\nu} \quad , \quad \tilde{F}_{\mu\nu\xi} = F_{\mu\nu\xi y} \tag{A.3}$$

while for IIB to IIA they are

$$\tilde{F}_{\mu\nu\sigma\xi\kappa y} = F_{\mu\nu\sigma\xi\kappa} \quad , \quad \tilde{F}_{\mu\nu\sigma\xi} = F_{\mu\nu\sigma\xi y} \quad , \quad \tilde{F}_{\mu\nu\sigma y} = F_{\mu\nu\sigma} \quad , \quad \tilde{F}_{\mu\nu} = -F_{\mu\nu y} \tag{A.4}$$

with $\mu, \nu, \sigma, \xi, \kappa \neq y$. Note that we have the convention that $F^{\mu_1 \cdots \mu_4} = \sqrt{-g}\epsilon^{\mu_1 \cdots \mu_{10}} F_{\mu_5 \cdots \mu_{10}}$ with $\epsilon^{01 \cdots 9} = 1$.

M/IIA S-duality conventions

Consider an M-theory background with the coordinate u being an explicit space-like isometry with $g_{uu} = 1$. We consider the S-duality between type IIA string theory and M-theory with u being the eleventh direction. The relation between the eleven- and ten-dimensional metrics is

$$ds^2_{\mathrm{M}} = ds^2_{\mathrm{IIA}} + (du + A_\mu dx^\mu)^2 \tag{A.5}$$

where A_μ is the one-form RR gauge potential in type IIA string theory. The dilaton is constant since we assumed $g_{uu} = 1$. The relations between the four-form field strength in M-theory and the RR four-form and NS-NS three-form field strength in type IIA are

$$F^{(\mathrm{M})}_{\mu\nu\sigma u} = H_{\mu\nu\sigma} \quad , \quad F^{(\mathrm{M})}_{\mu\nu\sigma\xi} = F^{(\mathrm{IIA})}_{\mu\nu\sigma\xi} \tag{A.6}$$

B Computations of supersymmetry

We compute the supersymmetry of various solutions found in Section 2.

B.1 Supersymmetry of compactified pp-waves

We consider here the supersymmetry of pp-waves of type IIB which have a compact direction. We follow the discussion of [4, 16, 41, 20]. In general the supersymmetry variation of the dilatino λ and the gravitino ψ_μ can be written[14]

$$\delta\lambda = \Gamma^+ W \eta \quad , \quad \delta\psi_\mu = D_\mu \eta + \Omega_\mu \eta \tag{B.1}$$

[14] In this section we use the notation and conventions of [20] so the definition of Γ^μ here is different from the rest of the paper. For example

$$\Gamma^3 \sigma_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Gamma^3 \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \Gamma^3 \psi_2 \\ \Gamma^3 \psi_1 \end{pmatrix}$$

and so forth. Here ψ_1,ψ_2 are two Majorana spinors with same chirality both with 16 components. We also use that $\Gamma^+ = \Gamma^0 + \Gamma^9$, $\Gamma^- = \frac{1}{2}(\Gamma^0 - \Gamma^9)$ and $\Gamma^{12345678} = \Gamma^{09}$.

where Ω_μ is the torsion in type IIB. W and Ω_μ involves both NS-NS and RR field strengths. We use the conventions of [20] for the supersymmetry variations (B.1) in type IIB. We then compactify the solutions along the Killing vector with unit norm

$$\xi = \partial_+ - \frac{1}{2}\partial_- + \beta^2 \sum_{i<j} J_{ij} M_{ij} \tag{B.2}$$

where $M_{ij} = \tilde{x}^i \partial_j - \tilde{x}^j \partial_i$. This leads to the extra condition on Killing spinors

$$Q\eta = 0 \quad , \qquad Q \equiv \Omega_+ + \frac{1}{2}\sum_{i<j} J_{ij}\Gamma^{ij} \tag{B.3}$$

where we have put $\beta = 1$ for simplicity. To further simplify the analysis we define

$$J_1 = i\sigma_2 \Gamma^{12} \quad , \qquad J_2 = i\sigma_2 \Gamma^{34} \quad , \qquad J_3 = i\sigma_2 \Gamma^{56} \quad , \qquad J_4 = i\sigma_2 \Gamma^{78} \tag{B.4}$$

We see that the eigenvalues of J_i are ± 1. Note that $J_1 J_2 J_3 J_4$ has eigenvalue 1 for kinematical supersymmetries, i.e. when $\Gamma^+ \eta = 0$, and eigenvalue -1 for dynamical supersymmetries, i.e. when $\Gamma^- \eta = 0$. Note also that any eigenvalue of (J_1, J_2, J_3, J_4) has degeneracy two.

We consider now the maximally supersymmetric pp-wave of type IIB (2.53)-(2.54) [4] with 32 supersymmetries. We have

$$\begin{aligned}
\Omega_+ &= -\frac{1}{4}(\Gamma^{1234} + \Gamma^{5678})\Gamma^+\Gamma^-(i\sigma_2) = i\sigma_2 \frac{1}{4}(J_1 J_2 + J_3 J_4)(1 + J_1 J_2 J_3 J_4) \\
Q &= i\sigma_2 \left[\frac{1}{4}(J_1 J_2 + J_3 J_4)(1 + J_1 J_2 J_3 J_4) - \frac{1}{2}(J_1 + J_2 + J_3 - J_4) \right]
\end{aligned} \tag{B.5}$$

When the direction (B.2) is compact, we see that we have 8 kinematical and 12 dynamical supersymmetries, giving altogether 20 supersymmetries.

For the mixed pp-wave solution with NS-NS and RR three-form flux (2.38), (2.49) [6] we compute

$$\begin{aligned}
W &= -\frac{1}{2}\Big[\sigma_1 \sin\gamma + \sigma_3 \cos\gamma\Big](J_1 + J_2) \\
\Omega_+ &= -\frac{1}{4}\Big[2\sigma_1 \sin\gamma - (1 + J_1 J_2 J_3 J_4)\sigma_3 \cos\gamma\Big](J_1 + J_2) \\
Q &= -\frac{1}{4}\Big[2\sigma_1 \sin\gamma - (1 + J_1 J_2 J_3 J_4)\sigma_3 \cos\gamma - 2i\sigma_2\Big](J_1 + J_2)
\end{aligned} \tag{B.6}$$

We see that the dilatino variation $\Gamma^+ W \eta = 0$ gives 24 supersymmetries in the uncompactified case, i.e. 16 kinematical and 8 dynamical. When the direction (B.2) is compact we see that we need $J_1 + J_2 = 0$ for a generic value of γ, thus giving 16 supersymmetries, i.e. 8 kinematical and 8 dynamical. If $\gamma = 0$ then we get instead 12 kinematical and 8 dynamical supersymmetries, giving altogether 20 supersymmetries.

For the mixed pp-wave solution with RR three and five-form flux (2.61)-(2.62) [20] we have

$$
\begin{aligned}
W &= -\frac{1}{2}\sin\gamma\,\sigma_3(J_1+J_2+J_3-J_4) \\
\Omega_+ &= \frac{1}{4}i\sigma_2\Big[(J_1J_2+J_3J_4)\cos\gamma-(J_1+J_2+J_3-J_4)\sigma_1\sin\gamma\Big](1+J_1J_2J_3J_4) \\
Q &= i\sigma_2\left[\frac{1}{4}\Big((J_1J_2+J_3J_4)\cos\gamma-(J_1+J_2+J_3-J_4)\sigma_1\sin\gamma\Big)(1+J_1J_2J_3J_4)\right. \\
&\qquad \left.-\frac{1}{2}(J_1+J_2+J_3-J_4)\right]
\end{aligned}
\tag{B.7}
$$

We see that the dilatino variation $\Gamma^+W\eta=0$ gives 28 supersymmetries in the uncompactified case, i.e. 16 kinematical and 12 dynamical. When the direction (B.2) is compact we see that we have 0 kinematical and 12 dynamical supersymmetries, giving altogether 12 supersymmetries. It is also easy to see that the special case $\gamma=0,\pi/2$ there are 20 supersymmetries (12 dynamical ones + 8 kinematical ones).

B.2 Supersymmetry of M-theory backgrounds

Here we show the analysis of spinors of several Gödel backgrounds in M-theory explicitly.
(1) M-theory Gödel background (2.57) and (2.58) (20 susy)

After we solve the Killing spinor equations (2.36), we obtain the following ten constant spinors

$$
\eta^{(1)}=\eta^{--++},\quad \eta^{(2)}=\eta^{-++-},\quad \eta^{(3)}=\eta^{+-+-}\quad \eta^{(4)}=\eta^{--+-},\quad \eta^{(5)}=\eta^{----}, \tag{B.8}
$$

where we specify by using the ±1 values of $(\gamma^{012},\gamma^{034},\gamma^{056},\gamma^{078})$ (note that there is the degeneracy of factor two) as well as the other ten spinors which depend on the coordinates $x^1,...,x^8$ linearly

$$
\begin{aligned}
\eta^{(6)} &= \left(1+2\beta J_{ij}\Gamma^{0i}x^j\right)\eta^{++-+}, \\
\eta^{(7)} &= \left(1-2\beta\Gamma^{04}x^3+2\beta\Gamma^{03}x^4-2\beta\Gamma^{06}x^5+2\beta\Gamma^{05}x^6\right)\eta^{+-++}, \\
\eta^{(8)} &= \left(1-2\beta\Gamma^{02}x^1+2\beta\Gamma^{01}x^2-2\beta\Gamma^{06}x^5+2\beta\Gamma^{05}x^6\right)\eta^{-+++}, \\
\eta^{(9)} &= \left(1-2\beta\Gamma^{04}x^3+2\beta\Gamma^{03}x^4+2\beta\Gamma^{08}x^7-2\beta\Gamma^{07}x^8\right)\eta^{+---}, \\
\eta^{(10)} &= \left(1-2\beta\Gamma^{02}x^1+2\beta\Gamma^{01}x^2+2\beta\Gamma^{08}x^7+2\beta\Gamma^{07}x^8\right)\eta^{-+--}.
\end{aligned}
\tag{B.9}
$$

Thus we can conclude that there are 20 supersymmetries in this background.

(2) M-theory Gödel background (2.81),(2.82) and (2.83) (18 or 16 susy)

In this case again we have only to solve (2.36). We find the constant 10 Killing spinors

$$
\chi^{(1)}=\chi^{+-++},\quad \chi^{(2)}=\chi^{-+++},\quad \chi^{(3)}=\chi^{--++},\quad \chi^{(4)}=\chi^{--+-},\quad \chi^{(5)}=\chi^{---+}, \tag{B.10}
$$

and the non-constant 6 Killing spinors

$$
\begin{aligned}
\chi^{(6)} &= \left(1+2\beta J_{ij}\Gamma^{0i}x^j\right)\chi^{++--}, \\
\chi^{(7)} &= \left(1-2k\beta\Gamma^{04}x^3+2k\beta\Gamma^{03}x^4-2k\beta\Gamma^{06}x^5+2k\beta\Gamma^{05}x^6\right)\chi^{+--+}, \\
\chi^{(8)} &= \left(1-2k\beta\Gamma^{04}x^3+2k\beta\Gamma^{03}x^4-2k\beta\Gamma^{08}x^7+2k\beta\Gamma^{07}x^8\right)\chi^{+-+-},
\end{aligned}
\tag{B.11}
$$

where the values of $(J_{12}, J_{34}, J_{56}, J_{78}, J_{9\sharp})$ is given by $(k+1, k, k, k, 1)$ as defined in (2.83). Thus we can conclude that there are 16 supersymmetries in this model.

If we assume $k = 1$, we get the extra Killing spinor

$$\chi^{(9)} = \left(1 - 2\beta\Gamma^{04}x^3 + 2\beta\Gamma^{03}x^4 + 2\beta\Gamma^{0\sharp}x^9 - 2\beta\Gamma^{09}x^\sharp\right)\chi^{+---}. \tag{B.12}$$

Therefore in this special case $k = 1$ we get 18 supersymmetries.

(3) M-theory Gödel background (2.65)-(2.66) (12 susy)

Since we have already shown that the small radius limit (type IIA model) has 12 supersymmetries (see (B.7)) we have only to check that the background in M-theory does not allow more than 12 supersymmetries. To see this let us consider the time component $\mu = 0$ of Killing spinor equation (2.29). This is given by

$$\begin{aligned}\left(\partial_0 + \frac{\beta}{2}(\Gamma^{12} + \Gamma^{34} + \Gamma^{56} - \Gamma^{78})\right)\eta &= \frac{1}{2}\beta\cos\gamma(\Gamma^{01234} + \Gamma^{05678})\eta \\ -\frac{1}{6}\beta(\Gamma^{0129\sharp} + \Gamma^{0349\sharp} + \Gamma^{0569\sharp} - \Gamma^{0789\sharp})\eta &+ \frac{1}{3}\beta\sin\gamma(\Gamma^{129} + \Gamma^{349} + \Gamma^{569} - \Gamma^{789})\eta.\end{aligned} \tag{B.13}$$

For generic γ ($\gamma \neq 0, \pi/2$), we have the conditions $(\Gamma^{1234} + \Gamma^{5678})\eta = (\Gamma^{012} + \Gamma^{034} + \Gamma^{056} - \Gamma^{078})\eta = 0$. Thus we find 12 supersymmetries in this background.

(4) M-theory Gödel background (2.50)-(2.51) (16 susy)

The Killing spinor can be examined as in the previous case (B.13). The result is given by the constraint $(\Gamma^{12} + \Gamma^{34})\eta = 0$, and thus we have 16 supersymmetries generically.

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